

MULTI-MICROLOCALIZATION AND MICROSUPPORT

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Abstract

The purpose of this paper is to establish the foundations of multi-microlocalization, in particular, to give the fiber formula for the multi-microlocalization functor and estimate of microsupport of a multi-microlocalized object. We also give some applications of these results.

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Introduction

A microlocalized object of a sheaf F along a closed submanifold M is locally described by, roughly speaking, local cohomology groups of F with support in a dual cone of the edge M . As a result it can be tightly related with, via Čech cohomology groups, boundary values of local sections of F defined on open cones of the edge M . It is well known that, for example, Sato's microfunctions, which are obtained by applying the microlocalization functor along a real analytic manifold M to the sheaf of holomorphic functions, can be regarded as boundary values of holomorphic functions locally defined on wedges with the edge M .

We sometimes, in study of partial differential equations, need to consider a boundary value of a function defined on a cone along a family χ of several closed submanifolds. In such a study, J. M. Delort [1] had introduced *simultaneous microlocalization* along a normal crossing divisor χ which gives a boundary value of a function defined on a dual poly-sector. *Bi-microlocalization* along submanifolds $\chi = \{M_1, M_2\}$ with $M_1 \subset M_2$ was also introduced by P. Schapira and K. Takeuchi [14] and [15] which defines a different kind of a boundary value.

On the other hand, in the paper [4], the first and the second authors of this article established the notion of *the multi-normal cone* for a family χ of closed submanifolds with a suitable configuration, and they also constructed *the multi-specialization functor* along χ . We can observe that cones appearing in simultaneous microlocalization and bi-microlocalization are charac-

terized by using the multi-normal cone and that the both microlocalization functors coincide with the *multi-microlocalization functor* along χ where the latter functor is obtained by repeated application of Sato's Fourier transformation to the multi-specialization functor. Hence multi-microlocalization gives us a uniform machinery for the both simultaneous microlocalization and bi-microlocalization. The purpose of this paper is to establish the foundations of multi-microlocalization, in particular, to give the fiber formula for the multi-microlocalization functor and estimate of microsupport of a multi-microlocalized object. We briefly explain, in what follows, these two important results and their meanings.

The most fundamental question for the multi-microlocalization functor μ_χ along closed submanifolds $\chi = \{M_1, \dots, M_\ell\}$ is a shape of a cone on which a boundary value given by μ_χ is defined. The fiber formula gives us an explicit answer: A germ of $H^k(\mu_\chi(F))$ is isomorphic to local cohomology groups $\varinjlim_G H_G^k(F)$ where G is a vector sum of closed cones G_i 's and each G_i is defined in the similar way as that in the fiber formula of the usual microlocalization functor along M_i . Therefore the multi-microlocalization functor can be understood as a natural extension of the usual microlocalization functor. Once we have grasped a geometrical aspect of multi-microlocalization, then the next fundamental problem to be considered is estimate of microsupport of $\mu_\chi(F)$ by that of F , for which the answer is quite simple and beautiful: The $SS(\mu_\chi(F))$ is contained in the multi-normal cone of $SS(F)$ along χ^* . Here χ^* is a family of Lagrangian submanifolds $\{T_{M_1}^*X, \dots, T_{M_\ell}^*X\}$. This shows, in particular, soundness of our framework in the sense that the sharp estimate can be achieved by a geometrical tool (the multi-normal cone) already prepared in our framework. These two results have many applications, and some of them will be given in the last two sections of this paper.

The paper is organized as follows: We briefly recall, in Section 1, the theory of the multi-specialization developed in [4]. Then, in Section 2, we define the multi-microlocalization functor by repeatedly applying Sato's Fourier transformation to the multi-specialization functor. After showing several basic properties of the functor, we establish a fiber formula which explic-

itly describes a stalk of a multi-microlocalized object. By the fiber formula and an edge of the wedge theorem with bounds shown in Section 4, we can construct the sheaf of microfunctions along χ that is a natural extension of the sheaf of Sato's microfunctions. In section 3, after some geometrical preparations, we give estimate of microsupport of a microlocalized object, that is our main result. Several applications of this result to \mathcal{D} modules are studied in Section 5.

1 Multi-specialization: a review

In this section we recall some results of [4]. We first fix some notations, then we recall the notion of multi-normal deformation and the definition of the functor of multi-specialization with some basic properties.

1.1 Notations

Let X be a real analytic manifold with $\dim X = n$, and let $\chi = \{M_1, \dots, M_\ell\}$ be a family of closed submanifolds in X ($\ell \geq 1$). Throughout the paper all the manifolds are always assumed to be countable at infinity.

We set, for $N \in \chi$ and $p \in N$,

$$\mathrm{NR}_p(N) := \{M_j \in \chi; p \in M_j, N \not\subseteq M_j \text{ and } M_j \not\subseteq N\}.$$

Let us consider the following conditions for χ .

H1 Each $M_j \in \chi$ is connected and the submanifolds are mutually distinct, i.e. $M_j \neq M_{j'}$ for $j \neq j'$.

H2 For any $N \in \chi$ and $p \in N$ with $\mathrm{NR}_p(N) \neq \emptyset$, we have

$$(1.1) \quad \left(\bigcap_{M_j \in \mathrm{NR}_p(N)} T_p M_j \right) + T_p N = T_p X.$$

Note that, if χ satisfies the condition H2, the configuration of two submanifolds must be either 1. or 2. below.

1. Both submanifolds intersect transversely.
2. One of them contains the other.

If χ satisfies the condition H2, then for any $p \in X$, there exist a neighborhood V of p in X , a system of local coordinates (x_1, \dots, x_n) in V and a family of subsets $\{I_j\}_{j=1}^\ell$ of the set $\{1, 2, \dots, n\}$ for which the following conditions hold.

1. Either $I_k \subset I_j$, $I_j \subset I_k$ or $I_k \cap I_j = \emptyset$ holds ($k, j \in \{1, 2, \dots, \ell\}$).
2. A submanifold $M_j \in \chi$ with $p \in M_j$ ($j = 1, 2, \dots, \ell$) is defined by $\{x_i = 0; i \in I_j\}$ in V .

We set, for $N \in \chi$,

$$(1.2) \quad \iota_\chi(N) := \bigcap_{N \subsetneq M_j} M_j.$$

Here $\iota_\chi(N) := X$ for convention if there exists no j with $N \subsetneq M_j$. When there is no risk of confusion, we write for short $\iota(N)$ instead of $\iota_\chi(N)$. We also assume the condition H3 below for simplicity.

H3 $M_j \neq \iota(M_j)$ for any $j \in \{1, 2, \dots, \ell\}$.

In local coordinates let $I_1, \dots, I_\ell \subseteq \{1, \dots, n\}$ such that $M_i = \{x_k = 0; k \in I_i\}$. Note that the family χ satisfies the conditions H1, H2 and H3 if and only if I_1, \dots, I_ℓ satisfy the corresponding conditions

$$(1.3) \quad \begin{aligned} & \text{(i) either } I_j \subsetneq I_k, I_k \subsetneq I_j \text{ or } I_j \cap I_k = \emptyset \text{ holds for any } j \neq k, \\ & \text{(ii) } \left(\bigcup_{I_k \subsetneq I_j} I_k \right) \subsetneq I_j \text{ for any } j. \end{aligned}$$

Hence, for any $j \in \{1, 2, \dots, \ell\}$, the set

$$(1.4) \quad \hat{I}_j := I_j \setminus \left(\bigcup_{I_k \subsetneq I_j} I_k \right)$$

is not empty by the condition H3. Further it follows from the conditions H1, H2 and H3 that, for each $j \in \{1, \dots, \ell\}$, there exist unique $j_1, \dots, j_p \in \{1, \dots, \ell\}$ such that

$$I_j = \hat{I}_{j_1} \sqcup \dots \sqcup \hat{I}_{j_p}$$

and $I_k \subseteq I_j$ for all $k \in \{j_1, \dots, j_p\}$ (equality holds only if $k = j$). In particular

$$(1.5) \quad \bigcup_{1 \leq j \leq \ell} I_j = \hat{I}_1 \sqcup \dots \sqcup \hat{I}_\ell.$$

Set, for $i \in \{1, \dots, n\}$

$$(1.6) \quad J_i = \{j \in \{1, \dots, \ell\} ; i \in I_j\}.$$

It follows from the proof of Proposition 1.3 of [4] that

$$(1.7) \quad J_\alpha = J_\beta \Leftrightarrow \alpha, \beta \in \hat{I}_j$$

for some $j \in \{1, \dots, \ell\}$.

Lemma 1.1. *Let $I_i \supseteq I_j$. Then for each $\alpha \in \hat{I}_i$ and $\beta \in \hat{I}_j$ we have $J_\alpha \subseteq J_\beta$.*

Proof. Let $\alpha \in \hat{I}_i$ and $\beta \in \hat{I}_j$. By definition of \hat{I}_i and condition H2 we have

$$k \in J_\alpha \Leftrightarrow I_k \supseteq I_i.$$

Since $I_i \supseteq I_j$ we have

$$I_k \supseteq I_j \Leftrightarrow k \in J_\beta.$$

Then $J_\alpha \subseteq J_\beta$. □

Thanks to the previous result we can introduce the following notation. Set for convenience

$$(1.8) \quad I_0 = \hat{I}_0 := \{1, \dots, n\} \setminus \bigcup_{j=1}^{\ell} I_j.$$

Then, in local coordinates, we can write the coordinates (x_1, \dots, x_n) by

$$(1.9) \quad (x^{(0)}, x^{(1)}, \dots, x^{(\ell)}),$$

where $x^{(j)}$ denotes the coordinates $(x_i)_{i \in \hat{I}_j}$ ($j = 0, \dots, \ell$). By taking (1.7) into account, we can also define, for $j \in \{0, 1, \dots, \ell\}$

$$(1.10) \quad \hat{J}_j = \{k \in \{1, \dots, \ell\} ; \hat{I}_j \subseteq I_k\} = \{k \in \{1, \dots, \ell\} ; I_j \subseteq I_k\}.$$

Note that, with this notation, we have $\hat{J}_0 = \emptyset$. Moreover, by (1.7) we have $J_\alpha = J_\beta = \hat{J}_j$ for each $j \in \{1, \dots, \ell\}$ and each $\alpha, \beta \in \hat{I}_j$. In particular, by Lemma 1.1

$$(1.11) \quad I_i \subseteq I_j \Rightarrow \hat{J}_j \subseteq \hat{J}_i.$$

1.2 Multi-normal deformation

In [4] the notion of multi-normal deformation was introduced. Here we consider a slight generalization where we replace the condition H2 with the weaker one. Let $\chi = \{M_1, \dots, M_\ell\}$ be a family of closed submanifolds of X . We say that χ is *simultaneously linearizable* on $M = M_1 \cap \dots \cap M_\ell$ if for every $x \in M$ there exist a neighborhood V of x and a system of local coordinates (x_1, \dots, x_n) there for which we can find subsets I_j 's of $\{1, \dots, n\}$ such that each $M_j \cap V$ is defined by equations $x_i = 0$ ($i \in I_j$). Note that if χ satisfies the condition H2, then it is simultaneously linearizable. Now, through the section, we assume that χ is simultaneously linearizable on M .

First recall the classical construction of [7] of the normal deformation of X along M_1 . We denote it by \tilde{X}_{M_1} and we denote by $t_1 \in \mathbb{R}$ the deformation parameter. Let $\Omega_{M_1} = \{t_1 > 0\}$ and let us identify $s^{-1}(0)$ with $T_{M_1}X$. We have the commutative diagram

$$(1.12) \quad \begin{array}{ccc} T_{M_1}X & \xrightarrow{s_{M_1}} & \tilde{X}_{M_1} \xleftarrow{i_{\Omega_{M_1}}} \Omega_{M_1} \\ \downarrow \tau_{M_1} & & \downarrow p_{M_1} \swarrow \tilde{p}_{M_1} \\ M & \xrightarrow{i_{M_1}} & X. \end{array}$$

Set $\tilde{\Omega}_{M_1} = \{(x; t_1) ; t_1 \neq 0\}$ and define

$$\widetilde{M}_2 := \overline{(p_{M_1}|_{\tilde{\Omega}_{M_1}})^{-1}M_2}.$$

Then \widetilde{M}_2 is a closed smooth submanifold of \tilde{X}_{M_1} .

Remark 1.2. One cannot expect the smoothness of \widetilde{M}_2 without the simultaneously linearizable condition. For example, let $X = \mathbb{R}^2$ with (x_1, x_2) and let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_1 - x_2^2 = 0\}$. Then in \widetilde{X}_{M_1} with coordinates $(x_1, x_2; t_1)$ we have

$$\widetilde{M}_2 := \overline{\{t_1 x_1 - x_2^2 = 0, t_1 \neq 0\}} = \{t_1 x_1 - x_2^2 = 0\}$$

which is singular at $(0, 0; 0)$.

Now we can define the normal deformation along M_1, M_2 as

$$\widetilde{X}_{M_1, M_2} := (\widetilde{X}_{M_1})_{\widetilde{M}_2}.$$

Then we can define recursively the normal deformation along χ as

$$\widetilde{X} = \widetilde{X}_{M_1, \dots, M_\ell} := (\widetilde{X}_{M_1, \dots, M_{\ell-1}})_{\widetilde{M}_\ell}.$$

Set $S_\chi = \{t_1, \dots, t_\ell = 0\}$, $M = \bigcap_{i=1}^\ell M_i$ and $\Omega_\chi = \{t_1, \dots, t_\ell > 0\}$. Then we have the commutative diagram

$$(1.13) \quad \begin{array}{ccc} S_\chi & \xrightarrow{s} & \widetilde{X} \xleftarrow{i_\Omega} \Omega_\chi \\ \downarrow \tau & & \downarrow p \swarrow \tilde{p} \\ M & \xrightarrow{i_M} & X. \end{array}$$

Let us consider the diagram (1.13). In local coordinates let $I_1, \dots, I_\ell \subseteq \{1, \dots, n\}$ such that $M_i = \{x_k = 0; k \in I_i\}$. For $j \in \{0, \dots, \ell\}$ set

$$\hat{J}_j = \{k \in \{1, \dots, \ell\}; \hat{I}_j \subseteq I_k\}, \quad t_{\hat{J}_j} = \prod_{k \in \hat{J}_j} t_k,$$

where $t_1, \dots, t_\ell \in \mathbb{R}$ and $t_{\hat{J}_0} = 1$. Then $p : \widetilde{X} \rightarrow X$ is defined by

$$(x^{(0)}, x^{(1)}, \dots, x^{(\ell)}; t_1, \dots, t_\ell) \mapsto (t_{\hat{J}_0} x^{(0)}, t_{\hat{J}_1} x^{(1)}, \dots, t_{\hat{J}_\ell} x^{(\ell)}).$$

Definition 1.3. Let Z be a subset of X . The multi-normal cone to Z along χ is the set $C_\chi(Z) = \overline{\tilde{p}^{-1}(Z)} \cap S_\chi$.

Lemma 1.4. Let $p = (p^{(0)}, p^{(1)}, \dots, p^{(\ell)}; 0, \dots, 0) \in S_\chi$ and let $Z \subset X$. The following conditions are equivalent:

1. $p \in C_\chi(Z)$.
2. There exist sequences $\{(c_{1,m}, \dots, c_{\ell,m})\} \subset (\mathbb{R}^+)^{\ell}$ and $\{(q_m^{(0)}, q_m^{(1)}, \dots, q_m^{(\ell)})\} \subset Z$ such that $q_m^{(j)} c_{j,j} \rightarrow p^{(j)}$, $j = 0, 1, \dots, \ell$ and $c_{m,j} \rightarrow +\infty$, $j = 1, \dots, \ell$.

Proof. Let us prove 1. \Rightarrow 2. Since $p \in \overline{\tilde{p}^{-1}(Z)} \cap S_\chi$ there exist sequences $\{(p_m^{(0)}, p_m^{(1)}, \dots, p_m^{(\ell)})\} \subset Z$, $\{(t_{1,m}, \dots, t_{\ell,m})\} \subset (\mathbb{R}^+)^{\ell}$ such that

$$\begin{cases} t_{j,m} \rightarrow 0, & j = 1, \dots, \ell, \\ (p_m^{(0)}, p_m^{(1)}, \dots, p_m^{(\ell)}) \rightarrow (p^{(0)}, p^{(1)}, \dots, p^{(\ell)}), \\ (p^{(0)}, p^{(1)} t_{j_1,m}, \dots, p^{(\ell)} t_{j_\ell,m}) \in Z. \end{cases}$$

Set $t_{j,m}^{-1} = c_{j,m}$, $j = 1, \dots, \ell$ and $p_m^{(j)} t_{j,j,m} = q_m^{(j)}$, $j = 0, 1, \dots, \ell$. Then we have $\{(q_m^{(0)}, q_m^{(1)}, \dots, q_m^{(\ell)})\} \subset Z$, $q_m^{(j)} c_{j,j} \rightarrow p^{(j)}$, $j = 0, 1, \dots, \ell$ and $c_{m,j} \rightarrow +\infty$, $j = 1, \dots, \ell$.

Let us prove 2. \Rightarrow 1. Suppose that there exist sequences $\{(c_{1,m}, \dots, c_{\ell,m})\} \subset (\mathbb{R}^+)^{\ell}$ and $\{(q_m^{(0)}, q_m^{(1)}, \dots, q_m^{(\ell)})\} \subset Z$ such that $q_m^{(j)} c_{j,j} \rightarrow p^{(j)}$, $j = 0, 1, \dots, \ell$ and $c_{m,j} \rightarrow +\infty$, $j = 1, \dots, \ell$. Define $p_m^{(j)} = q_m^{(j)} c_{j,j}$, $j = 0, 1, \dots, \ell$ and $t_{j,m} = c_{m,j}^{-1}$. Then clearly

$$\begin{cases} t_{j,m} \rightarrow 0, & j = 1, \dots, \ell, \\ (p_m^{(0)}, p_m^{(1)}, \dots, p_m^{(\ell)}) \rightarrow (p^{(0)}, p^{(1)}, \dots, p^{(\ell)}), \\ (p^{(0)}, p^{(1)} t_{j_1,m}, \dots, p^{(\ell)} t_{j_\ell,m}) \in Z. \end{cases}$$

So $p \in C_\chi(Z)$. □

Let us consider the canonical map $T_{M_j} \iota(M_j) \rightarrow M_j \hookrightarrow X$, $j = 1, \dots, \ell$, we write for short

$$\times_{X, 1 \leq j \leq \ell} T_{M_j} \iota(M_j) := T_{M_1} \iota(M_1) \times_X T_{M_2} \iota(M_2) \times_X \dots \times_X T_{M_\ell} \iota(M_\ell).$$

When χ satisfies the conditions H1, H2 and H3 we have

$$(1.14) \quad S_\chi \simeq \times_{X, 1 \leq j \leq \ell} T_{M_j} \iota(M_j).$$

Remark 1.5. When χ satisfies conditions H1, H2 and H3, the zero-section S_χ becomes a vector bundle over M . However, in general, the simultaneously linearizable condition is not enough to assure the existence of a vector bundle structure on S_χ . The important exceptional case where χ does not satisfy H2 but S_χ has a vector bundle structure will be studied in § 3.1.

Example 1.6. Let us see two typical examples of multi-normal deformations in the complex case. Let $X = \mathbb{C}^2$ ($\simeq \mathbb{R}^4$ as a real manifold) with coordinates (z_1, z_2) .

1. (Majima) Let $\chi = \{M_1, M_2\}$ with $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1\}$, $I_2 = \{2\}$, $J_1 = \{1\}$, $J_2 = \{2\}$ (in \mathbb{R}^4 , if $z_1 = (x_1, x_2)$ and $z_2 = (x_3, x_4)$ we have $I_1 = \{1, 2\}$, $I_2 = \{3, 4\}$, $J_1 = J_2 = \{1\}$, $J_3 = J_4 = \{2\}$). The map $p : \tilde{X} \rightarrow X$ is defined by

$$(z_1, z_2; t_1, t_2) \mapsto (t_1 z_1, t_2 z_2).$$

Remark that the deformation is real though X is complex. In particular $t_1, t_2 \in \mathbb{R}$. We have $\iota(M_1) = \iota(M_2) = X$ and then the zero section S of \tilde{X} is isomorphic to $T_{M_1} X \times_X T_{M_2} X$.

2. (Takeuchi) Let $\chi = \{M_1, M_2\}$ with $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1, 2\}$, $I_2 = \{2\}$, $J_1 = \{1\}$, $J_2 = \{1, 2\}$ (in \mathbb{R}^4 , if $z_1 = (x_1, x_2)$ and $z_2 = (x_3, x_4)$ we have $I_1 = \{1, 2, 3, 4\}$, $I_2 = \{3, 4\}$, $J_1 = J_2 = \{1\}$, $J_3 = J_4 = \{1, 2\}$). The map $p : \tilde{X} \rightarrow X$ is defined by

$$(z_1, z_2; t_1, t_2) \mapsto (t_1 z_1, t_1 t_2 z_2).$$

We have $\iota(M_1) = M_2$, $\iota(M_2) = X$ and then the zero section S of \tilde{X} is isomorphic to $T_{M_1} M_2 \times_X T_{M_2} X$.

Example 1.7. Let us see three typical examples of multi-normal deformations in the real case. Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) .

1. (Majima) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then χ satisfies H1, H2 and H3. We have

$I_1 = \{1\}, I_2 = \{2\}, I_3 = \{3\}, J_1 = \{1\}, J_2 = \{2\}, J_3 = \{3\}$. The map $p : \tilde{X} \rightarrow X$ is defined by

$$(x_1, x_2, x_3; t_1, t_2, t_3) \mapsto (t_1 x_1, t_2 x_2, t_3 x_3).$$

We have $\iota(M_1) = \iota(M_2) = \iota(M_3) = X$ and then the zero section S of \tilde{X} is isomorphic to $T_{M_1} X \times_X T_{M_2} X \times_X T_{M_3} X$.

2. (Takeuchi) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{0\}, M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1, 2, 3\}, I_2 = \{2, 3\}, I_3 = \{3\}, J_1 = \{1\}, J_2 = \{1, 2\}, J_3 = \{1, 2, 3\}$. The map $p : \tilde{X} \rightarrow X$ is defined by

$$(x_1, x_2, x_3; t_1, t_2, t_3) \mapsto (t_1 x_1, t_1 t_2 x_2, t_1 t_2 t_3 x_3).$$

We have $\iota(M_1) = M_2, \iota(M_2) = M_3, \iota(M_3) = X$ and then the zero section S of \tilde{X} is isomorphic to $T_{M_1} M_2 \times_X T_{M_2} M_3 \times_X T_{M_3} X$.

3. (Mixed) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{0\}, M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1, 2, 3\}, I_2 = \{2\}, I_3 = \{3\}, J_1 = \{1\}, J_2 = \{1, 2\}, J_3 = \{1, 3\}$. The map $p : \tilde{X} \rightarrow X$ is defined by

$$(x_1, x_2, x_3; t_1, t_2, t_3) \mapsto (t_1 x_1, t_1 t_2 x_2, t_1 t_3 x_3).$$

We have $\iota(M_1) = M_2 \cap M_3, \iota(M_2) = \iota(M_3) = X$ and then the zero section S is isomorphic to $T_{M_1} (M_2 \cap M_3) \times_X T_{M_2} X \times_X T_{M_3} X$.

Let $q \in \bigcap_{1 \leq j \leq \ell} M_j$ and $p_j = (q; \xi_j)$ be a point in $T_{M_j} \iota(M_j)$ ($j = 1, 2, \dots, \ell$).

We set $p = p_1 \times_X \dots \times_X p_\ell \in \times_{X, 1 \leq j \leq \ell} T_{M_j} \iota(M_j)$, and $\tilde{p}_j = (q; \tilde{\xi}_j) \in T_{M_j} X$ denotes the image of the point p_j by the canonical embedding $T_{M_j} \iota(M_j) \hookrightarrow T_{M_j} X$. We denote by $\text{Cone}_{\chi, j}(p)$ ($j = 1, 2, \dots, \ell$) the set of open conic cones in $(T_{M_j} X)_q \simeq \mathbb{R}^{n - \dim M_j}$ that contain the point $\tilde{\xi}_j \in (T_{M_j} X)_q \simeq \mathbb{R}^{n - \dim M_j}$.

Definition 1.8. We say that an open set $G \subset (TX)_q$ is a multi-cone along χ with direction to $p \in \left(\times_{X, 1 \leq j \leq \ell} T_{M_j} \iota(M_j) \right)_q$ if G is written in the form

$$G = \bigcap_{1 \leq j \leq \ell} \pi_{j,q}^{-1}(G_j) \quad G_j \in \text{Cone}_{\chi,j}(p)$$

where $\pi_{j,q} : (TX)_q \rightarrow (T_{M_j}X)_q$ is the canonical projection. We denote by $\text{Cone}_{\chi}(p)$ the set of multi-cones along χ with direction to p .

For any $q \in X$, there exists an isomorphism $\psi : X \simeq (TX)_q$ near q and $\psi(q) = (q; 0)$ that satisfies $\psi(M_j) = (TM_j)_q$ for any $j = 1, \dots, \ell$.

Let Z be a subset of X . When χ satisfies H1, H2 and H3 we also have the following equivalence: $p \notin C_{\chi}(Z)$ if and only if there exist an open subset $\psi(q) \in U \subset (TX)_q$ and a multi-cone $G \in \text{Cone}_{\chi}(\psi_*(p))$ such that $\psi(Z) \cap G \cap U = \emptyset$ holds.

Example 1.9. We now give two examples of multi-cones in the complex case. Let $X = \mathbb{C}^2$ with coordinates (z_1, z_2) .

1. (Majima) Let $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then $\text{Cone}_{\chi}(p)$ for $p = (0, 0; 1, 1)$ is nothing but the set of multi sectors along $Z_1 \cup Z_2$ with their direction to $(1, 1)$.
2. (Takeuchi) Let $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. For $p = (0, 0; 1, 1) \in T_{M_1}M_2 \times_X T_{M_2}X$, it is easy to see that a cofinal set of $\text{Cone}_{\chi}(p)$ is, for example, given by the family of the sets

$$\{(\eta_1, \eta_2); |\eta_1| < \epsilon|\eta_2|, \eta_2 \in S\}_{S \ni 1, \epsilon > 0},$$

where S is a sector in \mathbb{C} containing the direction 1.

Example 1.10. We now give three examples of multi-cones in the real case. Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) .

1. (Majima) Let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. For $p = (0, 0, 0; 1, 1, 1) \in T_{M_1}X \times_X T_{M_2}X \times_X T_{M_3}X$, it is easy to see that $\text{Cone}_{\chi}(p) = \{(\mathbb{R}^+)^3\}$.

2. (Takeuchi) Let $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. For $p = (0, 0, 0; 1, 1, 1) \in T_{M_1} M_2 \times_X T_{M_2} M_3 \times_X T_{M_3} X$, it is easy to see that a cofinal set of $\text{Cone}_\chi(p)$ is, for example, given by the family of the sets

$$\{(\xi_1, \xi_2, \xi_3); |\xi_2| + |\xi_3| < \epsilon \xi_1, |\xi_3| < \epsilon \xi_2, \xi_3 > 0\}_{\epsilon > 0}.$$

3. (Mixed) Let $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. For $p = (0, 0, 0; 1, 1, 1) \in T_{M_1} (M_2 \cap M_3) \times_X T_{M_2} X \times_X T_{M_3} X$, a cofinal set of $\text{Cone}_\chi(p)$ is, for example, given by the family of the sets

$$\{(\xi_1, \xi_2, \xi_3); |\xi_2| + |\xi_3| < \epsilon \xi_1, \xi_2 > 0, \xi_3 > 0\}_{\epsilon > 0}.$$

This definition is also compatible with the restriction to a subfamily of χ . Namely, let $k \leq \ell$ and $K = \{j_1, \dots, j_k\}$ be a subset of $\{1, 2, \dots, \ell\}$. Set $\chi_K = \{M_{j_1}, \dots, M_{j_k}\}$ and $S_K := T_{M_{j_1}} \iota_\chi(M_{j_1}) \times_X \dots \times_X T_{M_{j_k}} \iota_\chi(M_{j_k}) \times_X M$. Let Z be a subset of X . Then we have

$$C_\chi(Z) \cap S_K = C_{\chi_K}(Z) \cap S_K.$$

In the following we will denote with the same symbol $C_{\chi_K}(Z)$ the normal cone with respect to χ_K and its inverse image via the map $\tilde{X} \rightarrow \tilde{X}_{M_{j_1}, \dots, M_{j_k}}$.

1.3 Multi-specialization

Let k be a field and denote by $\text{Mod}(k_{X_{sa}})$ (resp. $D^b(k_{X_{sa}})$) the category (resp. bounded derived category) of sheaves on the subanalytic site X_{sa} . For the theory of sheaves on subanalytic sites we refer to [9, 11]. For the theory of multi-specialization we refer to [4].

Let χ be a family of submanifolds satisfying H1, H2 and H3.

Definition 1.11. The multi-specialization along χ is the functor

$$\begin{aligned} \nu_\chi^{sa}: D^b(k_{X_{sa}}) &\rightarrow D^b(k_{S_{\chi sa}}), \\ F &\mapsto s^{-1} R\Gamma_{\Omega_\chi} p^{-1} F. \end{aligned}$$

Remark 1.12. We can give a description of the sections of the multi-specialization of $F \in D^b(k_{X_{sa}})$: let V be a conic subanalytic open subset of S_χ . Then:

$$H^j(V; \nu_M^{sa} F) \simeq \varinjlim_U H^j(U; F),$$

where U ranges through the family of open subanalytic subsets of X such that $C_\chi(X \setminus U) \cap V = \emptyset$. Let $p = (q; \xi) \in \times_{X, 1 \leq j \leq \ell} T_{M_j} \iota(M_j)$, let $B_\epsilon \subset (TX)_q$ be an open ball of radius $\epsilon > 0$ with its center at the origin and set

$$\text{Cone}_\chi(p, \epsilon) := \{G \cap B_\epsilon; G \in \text{Cone}_\chi(p)\}.$$

Applying the functor $\rho^{-1}: D^b(k_{S_{X_{sa}}}) \rightarrow D^b(k_{S_\chi})$ (see [11] for details) we can calculate the fibers at $p \in \times_{X, 1 \leq j \leq \ell} T_{M_j} \iota(M_j)$ which are given by

$$(\rho^{-1} H^j \nu_\chi^{sa} F)_p \simeq \varinjlim_W H^j(W; F),$$

where W ranges through the family $\text{Cone}_\chi(p, \epsilon)$ for $\epsilon > 0$. If there is no risk of confusion, in the rest of the paper we will use the notation

$$\nu_\chi := \rho^{-1} \nu_\chi^{sa} : D^b(k_{X_{sa}}) \rightarrow D^b(k_{S_\chi}).$$

Remark that, if $F \in \text{Mod}(k_X)$ we have $\nu_\chi R\rho_*(F) \simeq \nu_\chi(F)$, the multi-specialization of F defined with the classical Grothendieck operations.

2 Multi-microlocalization

In this section we introduce the functor of multi-microlocalization as the Fourier-Sato transform of multi-specialization. We then compute its stalks as inductive limits of section supported on convex subanalytic cones.

2.1 Definition

Now we are going to apply the Fourier-Sato transform to the multi-specialization. We refer to [7] for the classical Fourier-Sato transform and to [12] for its generalization to subanalytic sheaves. First, we need a general result: Let

$\tau_i: E_i \rightarrow Z$ ($1 \leq i \leq \ell$) be vector bundles over Z , and let E_i^* be the dual bundle of E_i . We denote by \wedge_i and \vee_i the Fourier-Sato and the inverse Fourier-Sato transformations on E_i respectively. Moreover we denote by \wedge_i^* and \vee_i^* the Fourier-Sato and the inverse Fourier-Sato transformations on E_i^* respectively. Recall that

$$G^{\wedge_i} = (G^{\vee_i})^a \otimes \omega_{E_i^*/Z}^{\otimes -1}.$$

Here $\omega_{E_i^*/Z}$ is the dualizing complex and $\omega_{E_i^*/Z}^{\otimes -1}$ its dual. Set $E := E_1 \times_Z \cdots \times_Z E_\ell$ and $E^* := E_1^* \times_Z \cdots \times_Z E_\ell^*$ for short. Let $\tau: E \rightarrow Z$ be the canonical projection. Set $P'_i := \{(\eta, \xi) \in E_i \times E_i^*; \langle \eta, \xi \rangle \leq 0\}$. Further set

$$P' := P'_1 \times_Z \cdots \times_Z P'_\ell, \quad P^+ := E \times_Z E^* \setminus P',$$

and denote by $p'_1: P' \rightarrow E$, $p'_2: P' \rightarrow E^*$, and $p_1^+: P^+ \rightarrow E_i$, $p_2^+: P^+ \rightarrow E^*$ the canonical projections respectively. Let F and G be a multi-conic object on E and E^* respectively. Then we set for short \wedge_E (resp. \vee_E^*) the composition of the Fourier-Sato transforms \wedge_i (resp. the composition of the inverse Fourier-Sato transforms \vee_i^*) on E_i for each $i \in \{1, \dots, \ell\}$.

Remark 2.1. Let X, Y be two real analytic manifolds, and $f: Y \rightarrow X$ a real analytic mapping. We have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow \rho & & \downarrow \rho \\ Y_{\text{sa}} & \xrightarrow{f} & X_{\text{sa}} \end{array}$$

For subanalytic sheaves we can also define the functor of proper direct image $f_{!!}$, and its derived functor $Rf_{!!}$. Note that $Rf_{!!} \circ R\rho_* \not\cong R\rho_* \circ Rf_!$ in general. As in the notation above, let $i: Z \rightarrow E$ be the zero-section embedding. Then we have

$$R\tau_{!!} \circ R\rho_* = i^! \circ R\rho_* = R\rho_* \circ i^! = R\rho_* \circ R\tau_!.$$

Hence in what follows, we identify $R\tau_{!!}$ with $R\tau_!$.

Proposition 2.2. *Let F and G be multi-conic objects on E and E^* respectively.*

(1) F^{\wedge_E} and $G^{\vee_E^*}$ are independent of the order of the the Fourier-Sato transformations \wedge_i and inverse the Fourier-Sato transformations \vee_i^* respectively.

(2) It follows that

$$G^{\vee_E^*} = Rp'_{1*}p'^!_2G.$$

Proof. (1) By induction on ℓ , we may assume that $\ell = 2$. We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & p'_2 & & \\
 & \nearrow p'_1 & P' & \xrightarrow[p'_{2,2}]{p'_2} & P'_1 \times_Z E_2^* \xrightarrow[p'_{1,2}]{} E^* \\
 & & \downarrow p'_{1,1} & \square & \downarrow p'_{1,1} \\
 E & \xleftarrow[p'_{2,1}]{} & E_1 \times_Z P'_2 & \xrightarrow[p'_{2,2}]{} & E_1 \times_Z E_2^*
 \end{array}$$

Then we have

$$\begin{aligned}
 (2.1) \quad G^{\vee_1^* \vee_2^*} &= Rp'_{2,1*}p'^!_{2,2}Rp'_{1,1*}p'^!_{1,2}G = Rp'_{2,1*}Rp'_{1,1*}p'^!_{2,2}p'^!_{1,2}G \\
 &= Rp'_{1*}p'^!_2G.
 \end{aligned}$$

For the same reason, we obtain

$$G^{\vee_1^* \vee_2^*} = Rp'_{1*}p'^!_2G = G^{\vee_2^* \vee_1^*}.$$

Therefore we have

$$\wedge_1^* \wedge_2^* = (\vee_2^* \vee_1^*)^{-1} = (\vee_1^* \vee_2^*)^{-1} = \wedge_2^* \wedge_1^*.$$

Replacing (E_1^*, E_2^*) with (E_1, E_2) , we obtain

$$\wedge_1 \wedge_2 = \wedge_2 \wedge_1.$$

(2) For $\ell = 2$, the result follows by (2.1). Next, assume that $\ell > 2$. Set $E' := \times_{Z, 2 \leq i \leq \ell} E_i$, $E'^* := \times_{Z, 2 \leq i \leq \ell} E'^*_i$, $P'_{E'} := \times_{Z, 2 \leq i \leq \ell} P'_i$, and $\vee_{E'}^*$ the composition of \vee_i^* for each $i \in \{2, \dots, \ell\}$. We have the following commutative

diagram:

$$\begin{array}{ccccc}
 & & p'_2 & & \\
 & \nearrow p'_1 & & \searrow & \\
 & P' & \xrightarrow{p'_{E',2}} & P'_1 \times_Z E'^* & \xrightarrow{p'_{1,2}} E^* \\
 & \downarrow p'_{1,1} & \square & \downarrow p'_{1,1} & \\
 E & \xleftarrow{p'_{E',1}} E_1 \times_Z P'_{E'} & \xrightarrow{p'_{E',2}} & E_1 \times_Z E'^* & \\
 & & & &
 \end{array}$$

Then by induction hypothesis, we have

$$(G^{\vee_1^*})^{\vee_{E'}} = Rp'_{E',1*} p'^!_{E',2} Rp'_{1,1*} p'^!_{1,2} G = Rp'_{E',1*} Rp'_{1,1*} p'^!_{E',2} p'^!_{1,2} G = Rp'_{1*} p'^!_2 G.$$

Therefore, the induction proceeds. \square

We shall need some notation. For a subset $K = \{i_1, \dots, i_k\} \subseteq \{1, \dots, \ell\}$, set $\chi_K := \{M_i; i \in K\}$, $S_i := T_{M_i} \iota(M_i) \times_X M$ ($j = 1, \dots, \ell$) and

$$S_K := T_{M_{i_1}} \iota(M_{i_1}) \times_X \cdots \times_X T_{M_{i_k}} \iota(M_{i_k}) = S_{i_1} \times_M \cdots \times_M S_{i_k}.$$

Let S_K^* be the dual of S_K :

$$S_K^* := T_{M_{i_1}}^* \iota(M_{i_1}) \times_X \cdots \times_X T_{M_{i_k}}^* \iota(M_{i_k}) = S_{i_1}^* \times_M \cdots \times_M S_{i_k}^*.$$

Given $C_{i_j} \subseteq S_{i_j}$, $j = 1, \dots, k$, we set for short $C_K := C_{i_1} \times_X \cdots \times_X C_{i_k} \subset S_K$. Define \wedge_K as the composition of the Fourier-Sato transformation \wedge_{i_k} on S_{i_k} for each $i_k \in K$.

Let $I, J \subseteq \{1, \dots, \ell\}$ be such that $I \sqcup J = \{1, \dots, \ell\}$. We still denote by $\pi: S_I \times_M S_J^* \rightarrow M$ the projection. We define the functor $\nu_{\chi_I}^{\text{sa}} \mu_{\chi_J}^{\text{sa}}$ by

$$\nu_{\chi_I}^{\text{sa}} \mu_{\chi_J}^{\text{sa}}: D^b(k_{X_{\text{sa}}}) \ni F \mapsto \nu_{\chi}^{\text{sa}}(F)^{\wedge_J} \in D^b(k_{(S_I \times_M S_J^*)_{\text{sa}}}).$$

By Proposition 2.2, this is well defined; that is, this definition does not depend on the order of the Fourier-Sato transformations. Composing with the functor ρ^{-1} , we set for short

$$\nu_{\chi_I} \mu_{\chi_J} := \rho^{-1} \nu_{\chi_I}^{\text{sa}} \mu_{\chi_J}^{\text{sa}}: D^b(k_{X_{\text{sa}}}) \rightarrow D^b(k_{S_I \times_M S_J^*}).$$

When $I = \emptyset$ we obtain the functor of multi-microlocalization: Set $\wedge := \wedge_{\{1, \dots, \ell\}}$ for short.

Definition 2.3. The multi-microlocalization along χ is the functor

$$\mu_\chi^{\text{sa}}: D^b(k_{X_{\text{sa}}}) \ni F \mapsto \nu_\chi^{\text{sa}}(F)^\wedge \in D^b(k_{S_{\chi^{\text{sa}}}^*}).$$

As above, we set for short

$$\mu_\chi := \rho^{-1} \mu_\chi^{\text{sa}}: D^b(k_{X_{\text{sa}}}) \rightarrow D^b(k_{S_\chi^*}).$$

2.2 Stalks

Let X be a real analytic manifold and consider a family of submanifolds $\chi = \{M_1, \dots, M_\ell\}$ satisfying H1, H2 and H3. Let $S = T_{M_1} \iota(M_1) \times_X \dots \times_X T_{M_\ell} \iota(M_\ell)$. Locally $p \in S$ is given by $p = p_1 \times \dots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)})$, with $\xi^{(k)} \in T_{M_k} \iota(M_k)$. Set $M = \bigcap_{j=1}^\ell M_j$. Let $\tau_j: T_{M_j} \iota(M_j) \hookrightarrow T_{M_j} X$ denote the canonical injection and let $\pi_j: S \rightarrow T_{M_j} \iota(M_j)$ be the canonical projection.

Lemma 2.4. *Let $F \in D^b(k_{X_{\text{sa}}})$.*

- (i) *Let A be a multi-conic closed subanalytic subset of S . Then $H_A^k(S; \nu_\chi^{\text{sa}} F) \simeq \varinjlim_{Z, U} H_Z^k(U; F)$, where U ranges through the family of open subanalytic neighborhoods of M and Z is a closed subanalytic subset such that $C_\chi(Z) \subset A$.*
- (ii) *Suppose that $A = \bigcap_{j=1}^\ell \pi_j^{-1} A_j$ with A_j being a closed conic subanalytic subset in $T_{M_j} \iota(M_j) \times_{M_j} M$. Then $H_A^k(S; \nu_\chi^{\text{sa}} F) \simeq \varinjlim_{Z, U} H_Z^k(U; F)$, where U ranges through the family of open subanalytic neighborhoods of M and $Z = Z_1 \cap \dots \cap Z_\ell$ with each Z_j being a closed subanalytic subset in X and $C_{M_j}(Z_j) \subset (T_{M_j} X \setminus \tau_j(T_{M_j} \iota(M_j) \setminus A_j))$.*

Proof. (i) We have the exact sequence

$$\dots \rightarrow H_A^k(S; \nu_\chi^{\text{sa}}(F)) \rightarrow H^k(S; \nu_\chi^{\text{sa}} F) \rightarrow H^k(S \setminus A; \nu_\chi^{\text{sa}} F) \rightarrow \dots$$

We have $H^k(S; \nu_\chi^{\text{sa}} F) \simeq \varinjlim_U H^k(U; F)$, where U ranges through the family of subanalytic neighborhoods of M . Moreover

$$H^k(S \setminus A; \nu_\chi^{\text{sa}} F) \simeq \varinjlim_W H^k(W; F),$$

where $W \in \text{Op}(X_{sa})$ is such that $C_\chi(X \setminus W) \cap (S \setminus A) = \emptyset$. Setting $Z = X \setminus W$ we obtain

$$H^k(S \setminus A; \nu_\chi^{sa} F) \simeq \varinjlim_{U, Z} H^k(U \setminus Z; F),$$

where U ranges through the family of subanalytic neighborhoods of M and Z is closed subanalytic such that $C_\chi(Z) \subset A$. Then the result follows thanks to the five lemma applied to the exact sequence

$$\cdots \rightarrow \varinjlim_{U, Z} H_Z^k(U; F) \rightarrow \varinjlim_U H^k(U; F) \rightarrow \varinjlim_{U, Z} H^k(U \setminus Z; F) \rightarrow \cdots$$

where U ranges through the family of subanalytic neighborhoods of M and Z is closed subanalytic such that $C_\chi(Z) \subset A$.

(ii) Let $A = \bigcap_{j=1}^\ell \pi_j^{-1} A_j$ with $A_j \subset T_{M_j} \iota(M_j) \times_{M_j} M$. Set for short $S_j := T_{M_j} \iota(M_j) \times_{M_j} M$. Then we have $S \setminus A = \bigcup_{j=1}^\ell \pi_j^{-1}(S_j \setminus A_j)$. Let $W \in \text{Op}(X_{sa})$ be such that $C_\chi(X \setminus W) \cap (S \setminus A) = \emptyset$. Then $W = \bigcup_{j=1}^\ell W_j$ with $C_\chi(X \setminus W_j) \cap \pi_j^{-1}(S_j \setminus A_j) = \emptyset$. Let us find W_1, \dots, W_ℓ . Let \widetilde{W}_j be an open neighborhood of $\pi_j^{-1}(S_j \setminus A_j)$ in the multi-normal deformation \widetilde{X} of X . Then by Proposition 4.6 of [4] we have $C_\chi(X \setminus \widetilde{p}(\widetilde{W}_j \cap \Omega)) \cap \pi_j^{-1}(S_j \setminus A_j) = \emptyset$. Set $W_j = \widetilde{p}(\widetilde{W}_j \cap \Omega) \cap W$. Up to shrink W we may suppose $W = \bigcup_{j=1}^\ell W_j$. We have

$$\begin{aligned} C_\chi(X \setminus W_j) \cap \pi_j^{-1}(S_j \setminus A_j) = \emptyset &\Leftrightarrow C_\chi(X \setminus W_j) \cap (S_j \setminus A_j) = \emptyset \\ &\Leftrightarrow C_{M_j}(X \setminus W_j) \cap \tau_j(S_j \setminus A_j) = \emptyset, \end{aligned}$$

where, for the second condition, $S_j \setminus A_j$ is regarded as a subset of

$$\{(q; \xi^{(1)}, \dots, \xi^{(\ell)}) \in S; q \in M, \xi^{(k)} = 0 (k \neq j)\} \subset S.$$

And the first equivalence follows from Lemma 4.2 of [4] and the second one from Corollary 4.3 of [4]. Setting $Z = X \setminus W$ and $Z_j = X \setminus W_j$, we obtain $Z = \bigcap_{j=1}^\ell Z_j$ with $C_{M_j}(Z_j) \cap \tau_j(S_j \setminus A_j) = \emptyset$ and the result follows. \square

Remark 2.5. We correct typo: By Lemma 6.1 in [4], for any $F \in \mathbf{D}^b(\mathbb{k}_{X_{sa}})$ there is a natural isomorphism

$$\nu_\chi^{sa}(F) = s^{-1} R\Gamma_{\Omega_\chi}(p^{-1}F) \simeq s^!(p^!F)_{\Omega_\chi}.$$

Proposition 2.6. *Assume that χ satisfies the conditions H1, H2 and H3.*

Let $\tau: S_\chi \simeq \times_{X, 1 \leq j \leq \ell} T_{M_j} \iota(M_j) \rightarrow M$ be the canonical projection, where $M := \bigcap_{j=1}^\ell M_j$. Then

$$(2.2) \quad \nu_\chi^{sa}(F)|_M \simeq R\tau_* \nu_\chi^{sa}(F) \simeq F|_M,$$

$$(2.3) \quad R\Gamma_M(\nu_\chi^{sa}(F)) \simeq R\tau_! \nu_\chi^{sa}(F) \simeq R\Gamma_M(F).$$

Proof. Let $k: M \rightarrow S_\chi$ be the zero-section embedding, and $i: M \rightarrow X$ the canonical embedding. Then

$$\begin{aligned} F|_M &= k^{-1} s^{-1} p^{-1} F \rightarrow k^{-1} s^{-1} Ri_{\Omega_\chi} i_{\Omega_\chi}^{-1} p^{-1} (F) \\ &\simeq k^{-1} s^{-1} R\Gamma_{\Omega_\chi}(p^{-1} F) \simeq \nu_\chi^{sa}(F)|_M, \\ R\tau_! \nu_\chi^{sa}(F) &= k^! \nu_\chi^{sa}(F) \simeq k^! s^! (p^! F)_{\Omega_\chi} \rightarrow k^! s^! p^! F = R\Gamma_M(F). \end{aligned}$$

These morphisms are isomorphisms by the stalk formulae. \square

Set $S^* := T_{M_1}^* \iota(M_1) \times_X \dots \times_X T_{M_\ell}^* \iota(M_\ell)$. Let $V = V_1 \times_X \dots \times_X V_\ell$ be a multi-conic open subanalytic subset in S^* , and let $\pi: S^* \rightarrow M$ denote the canonical projection. We set, for short, $V^\circ := V_1^\circ \times_X \dots \times_X V_\ell^\circ$ the multi-polar cone in S .

Lemma 2.7. *Let $V = V_1 \times_X \dots \times_X V_\ell$ be a multi-conic open subanalytic subset in S^* such that $V \cap \pi^{-1}(q)$ is convex in S_q^* for $q \in \pi(V)$. Then $H^k(V; \mu_\chi^{sa} F) \simeq \varinjlim_{Z, U} H_Z^k(U; F)$, where U ranges through the family of open subanalytic subsets in X with $U \cap M = \pi(V)$ and $Z = Z_1 \cap \dots \cap Z_\ell$ with $C_{M_j}(Z_j) \subset (T_{M_j} X \setminus \tau_j(\text{Int}(V_j^{\circ a})))$. Here $(\cdot)^a$ denotes the inverse image of the antipodal map.*

Proof. Since the fiber of V_j is convex for each j the Fourier-Sato transform gives $H^k(V; \mu_\chi^{sa} F) \simeq H_{V^\circ}^k(S; \nu_\chi^{sa} F)$. Then the result follows from Lemma 2.4 (ii). \square

Let $p = p_1 \times \dots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)}) \in S^*$. For any $p_k \in T_{M_k}^* \iota(M_k)$, we define the subset in $(T_{M_k} X)_q$

$$(2.4) \quad p_k^\# := (T_{M_k} X)_q \setminus \tau_k(p_k^{\circ a}).$$

Here the subset $p_k^{\circ a}$ in $(T_{M_k} \iota(M_k))_q$ denotes the antipodal polar set of the point p_k , i.e., $p_k^{\circ a} = \{\eta \in (T_{M_k} \iota(M_k))_q; \langle \eta, \xi^{(k)} \rangle \leq 0\}$. Note that $p_k^\#$ is an open subset. Set for short $\mu_\chi := \rho^{-1} \mu_\chi^{sa}$. As a consequence of Lemma 2.7 we have

Corollary 2.8. *Let $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)}) \in S^*$, and let $F \in D^b(k_{X_{sa}})$. Then $H^k(\mu_\chi F)_p \simeq \varinjlim_{Z, U} H_Z^k(U; F)$, where $U \in \text{Op}(X_{sa})$ ranges through the family of open subanalytic neighborhoods of q and Z runs through a family of closed sets in the form $Z_1 \cap Z_2 \cap \cdots \cap Z_\ell$ with each Z_k being closed subanalytic in X and $C_{M_k}(Z_k)_q \subset p_k^\# \cup \{0\}$ ($k = 1, 2, \dots, \ell$).*

Now we are going to find a stalk formula for multi-microlocalization given by a limit of sections with support (locally) contained in closed convex cones. As the problem is local, we may assume that $X = \mathbb{R}^n$ and $q = 0$ with coordinates (x_1, \dots, x_n) , and that there exists a subset I_k ($k = 1, 2, \dots, \ell$) in $\{1, 2, \dots, n\}$ with the conditions (1.3) such that each submanifold M_k is given by $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n; x_i = 0 (i \in I_k)\}$. Recall that \hat{I}_k was defined by (1.4) and that we set $M = \cap_k M_k$ and $n_k = \sharp \hat{I}_k$. Then locally we have

$$X = M \times (N_1 \times N_2 \times \cdots \times N_\ell) = M \times N,$$

where N_k is \mathbb{R}^{n_k} with coordinates $x^{(k)} = (x_i)_{i \in \hat{I}_k}$. Set, for $k \in \{1, \dots, \ell\}$,

$$\begin{aligned} J_{\prec k} &:= \{j \in \{1, \dots, \ell\}, I_j \subsetneq I_k\}, \\ J_{\succ k} &:= \{j \in \{1, \dots, \ell\}, I_j \supsetneq I_k\}, \\ J_{\nparallel k} &:= \{j \in \{1, \dots, \ell\}, I_j \cap I_k = \emptyset\}. \end{aligned} \tag{2.5}$$

Clearly we have

$$k \in J_{\prec j} \Leftrightarrow I_k \subsetneq I_j \Leftrightarrow j \in J_{\succ k}, \tag{2.6}$$

and, by the conditions H1, H2 and H3, we also have

$$J_{\prec k} \sqcup \{k\} \sqcup J_{\succ k} \sqcup J_{\nparallel k} = \{1, 2, \dots, \ell\}. \tag{2.7}$$

Let $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)}) \in T_{M_1}^* \iota(M_1) \times_X \cdots \times_X T_{M_\ell}^* \iota(M_\ell)$ and consider the following conic subset in N

$$(2.8) \quad \gamma_k := \left\{ \begin{array}{ll} x^{(j)} = 0, & j \in J_{\prec k} \sqcup J_{\nparallel k} \\ (x^{(j)})_{j=1, \dots, \ell} \in N; \quad x^{(j)} \in \mathbb{R}^{n_j}, & j \in J_{\succ k} \\ \langle x^{(j)}, \xi^{(k)} \rangle > 0, & j = k \end{array} \right\}.$$

Note that, if $\xi^{(k)} = 0$, then γ_k is empty.

Example 2.9. We now compute γ_k of (2.8) on the complex case in the following two typical situations. Let $X = \mathbb{C}^2$ with coordinates (z_1, z_2) .

1. (Majima) Let $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then

$$\begin{aligned} \gamma_1 &= \{(z_1, 0); \operatorname{Re}\langle z_1, \eta_1 \rangle > 0\}, \\ \gamma_2 &= \{(0, z_2); \operatorname{Re}\langle z_2, \eta_2 \rangle > 0\}. \end{aligned}$$

2. (Takeuchi) Let $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. Then

$$\begin{aligned} \gamma_1 &= \{(z_1, 0); \operatorname{Re}\langle z_1, \eta_1 \rangle > 0\}, \\ \gamma_2 &= \{(z_1, z_2); \operatorname{Re}\langle z_2, \eta_2 \rangle > 0\}. \end{aligned}$$

Example 2.10. We now compute γ_k of (2.8) on the real case in three typical situations. Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) .

1. (Majima) Let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then

$$\begin{aligned} \gamma_1 &= \{(x_1, 0, 0); \langle x_1, \xi_1 \rangle > 0\}, \\ \gamma_2 &= \{(0, x_2, 0); \langle x_2, \xi_2 \rangle > 0\}, \\ \gamma_3 &= \{(0, 0, x_3); \langle x_3, \xi_3 \rangle > 0\}. \end{aligned}$$

2. (Takeuchi) Let $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then

$$\begin{aligned} \gamma_1 &= \{(x_1, 0, 0); \langle x_1, \xi_1 \rangle > 0\}, \\ \gamma_2 &= \{(x_1, x_2, 0); \langle x_2, \xi_2 \rangle > 0\}, \\ \gamma_3 &= \{(x_1, x_2, x_3); \langle x_3, \xi_3 \rangle > 0\}. \end{aligned}$$

3. (Mixed) Let $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then

$$\begin{aligned}\gamma_1 &= \{(x_1, 0, 0); \langle x_1, \xi_1 \rangle > 0\}, \\ \gamma_2 &= \{(x_1, x_2, 0); \langle x_2, \xi_2 \rangle > 0\}, \\ \gamma_3 &= \{(x_1, 0, x_3); \langle x_3, \xi_3 \rangle > 0\}.\end{aligned}$$

Theorem 2.11. *Let $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)}) \in S^*$, and let $F \in D^b(k_{X_{sa}})$. Then we have*

$$(2.9) \quad H^k(\mu_\chi F)_p \simeq \varinjlim_{G, U} H_G^k(U; F).$$

Here U is an open subanalytic neighborhood of q in X and G is a closed subanalytic subset in the form $M \times \left(\sum_{k=1}^{\ell} G_k \right)$ with G_k being a closed subanalytic convex cone in N satisfying $G_k \setminus \{0\} \subset \gamma_k$, where γ_k is defined in (2.8).

Proof. As a point of the base manifold M is irrelevant in the subsequent arguments, we may assume $M = \{0\}$ for simplicity. We also assume $q = 0$ and $|\xi^{(k)}| \leq 1$ for $k = 1, 2, \dots, \ell$.

We first prove that, for any $Z = Z_1 \cap \cdots \cap Z_\ell$ with Z_k being closed in X and $C_{M_k}(Z_k)_q \subset p_k^\# \cup \{0\}$ ($k = 1, 2, \dots, \ell$), there exists G described in the theorem with $G \supset Z$. As G_k is convex and it contains the origin, we have

$$G_1^\circ \cap \cdots \cap G_\ell^\circ = (G_1 + \cdots + G_\ell)^\circ$$

and

$$(G_1^\circ \cap \cdots \cap G_\ell^\circ)^\circ = \overline{G_1 + \cdots + G_\ell}$$

where G_k° designates the usual polar cone of G_k in X as a vector space. We shall find a closed convex cone V such that $V = V_1 \cap \cdots \cap V_\ell$ with V_k convex for each $k = 1, \dots, \ell$ and $V_k^\circ \setminus \{0\} \subset \gamma_k$ satisfying $V^\circ \supseteq Z$. Furthermore we choose V_k so that every V_k° is proper with respect to the same direction $\tilde{\xi} \neq 0$, i.e., $V^\circ \setminus \{0\} \subset \{x \in X; \langle x, \tilde{\xi} \rangle > 0\}$. In this way

$$V^\circ = (V_1^{\circ\circ} \cap \cdots \cap V_\ell^{\circ\circ})^\circ = \overline{V_1^\circ + \cdots + V_\ell^\circ} = V_1^\circ + \cdots + V_\ell^\circ$$

and setting $G_k := V_k^\circ$ we obtain the claim. Here the last equality follows from the fact that every V° is closed and properly contained in the same half space in X .

It follows from the definition of Z_k that there exists $\epsilon > 0$ and a closed convex cone $\Gamma_k \subset N_k$ with $\Gamma_k \setminus \{0\} \subset \{\langle x^{(k)}, \xi^{(k)} \rangle > 0\}$ which satisfies

$$Z_k \subset \{x \in X; x^{(k)} \in \Gamma_k\} \cup \{x \in X; \epsilon |x^{(k)}| \leq \sum_{j \in J_{\prec k}} |x^{(j)}|\}.$$

Note that, for k with $\xi^{(k)} = 0$, we always take $\Gamma_k = \{0\}$. The existence of such an ϵ and a Γ_k is shown in the following way. We set

$$N' := \times_{j \in J_{\prec k}} N_j, \quad N'' := \times_{j \in J_{\succ k} \cup J_{\nparallel k}} N_j,$$

for which we have $X = N' \times N_k \times N''$ with coordinates $(x', x^{(k)}, x'')$. Note that $M_k = \{0\}_{N' \times N_k} \times N''$ holds. We also define a closed subset D in N_k by $\{x^{(k)} \in N_k; \langle x^{(k)}, \xi^{(k)} \rangle \leq 0\}$. Then $C_{M_k}(Z_k)_q \subset p_k^\# \cup \{0\}$ implies that, for any $\theta \in D \setminus \{0\}$, there exist an open cone Q_θ in $N' \times N_k$ with direction $(0_{N'}, \theta)$ and an open neighborhood U_θ of q in X satisfying

$$(Q_\theta \times N'') \cap U_\theta \subset X \setminus Z_k.$$

Then, as $\{0\}_{N'} \times D$ is a closed conic subset in $N' \times N_k$, we can find a finite subset Θ in $D \setminus \{0\}$ such that

$$\begin{aligned} \{0\}_{N'} \times (D \setminus \{0\}) &\subset \bigcup_{\theta \in \Theta} Q_\theta, \\ \left(\left(\bigcup_{\theta \in \Theta} Q_\theta \right) \times N'' \right) \cap \left(\bigcap_{\theta \in \Theta} U_\theta \right) &\subset X \setminus Z_k. \end{aligned}$$

Then, by taking U in (2.9) sufficiently small so that $U \subset \bigcap_{\theta \in \Theta} U_\theta$, we may assume, from the beginning,

$$\left(\bigcup_{\theta \in \Theta} Q_\theta \right) \times N'' \subset X \setminus Z_k.$$

As $\bigcup_{\theta \in \Theta} Q_\theta$ is an open conic neighborhood of $\{0\}_{N'} \times (D \setminus \{0\})$ in $N' \times N_k$,

there exist an open cone T in N_k with $D \setminus \{0\} \subset T$ and $\epsilon > 0$ satisfying

$$\left\{ (x', x^{(k)}) \in N' \times N_k; x^{(k)} \in T, \sum_{j \in J_{\prec k}} |x^{(j)}| < \epsilon |x^{(k)}| \right\} \subset \bigcup_{\theta \in \Theta} Q_\theta.$$

Hence we have

$$\left\{ (x', x^{(k)}) \in N' \times N_k; x^{(k)} \in T, \sum_{j \in J_{\prec k}} |x^{(j)}| < \epsilon |x^{(k)}| \right\} \times N'' \subset X \setminus Z_k,$$

which is equivalent to saying that

$$\left\{ x \in X; x^{(k)} \in (N_k \setminus T) \right\} \cup \left\{ x \in X; \sum_{j \in J_{\prec k}} |x^{(j)}| \geq \epsilon |x^{(k)}| \right\} \supset Z_k.$$

This shows the existence of $\epsilon > 0$ and $\Gamma_k := N_k \setminus T$.

Now we set

$$Z_{k,\Gamma} := \left\{ x \in X; x^{(k)} \in \Gamma_k \right\}$$

$$Z_{k,\epsilon} := \left\{ x \in X; \epsilon |x^{(k)}| \leq \sum_{j \in J_{\prec k}} |x^{(j)}| \right\}.$$

Note that, for $k \in \{1, 2, \dots, \ell\}$ with $\hat{I}_k = I_k$, we have $Z_k \subset Z_{k,\Gamma}$ and no $Z_{k,\epsilon}$ appears.

Define $V = V_1 \cap \dots \cap V_\ell$. Here each V_k is given by, if $\xi^{(k)} \neq 0$,

$$\left\{ x \in X; x^{(k)} \in T_k, \delta |\langle x^{(k)}, \xi^{(k)} \rangle| \geq \sum_{j \in J_{\succ k}} |x^{(j)}| \right\}$$

where $\delta > 0$ and T_k is a proper closed convex cone in N_k with $T_k \subset \{x^{(k)} \in N_k; \langle x^{(k)}, \xi^{(k)} \rangle \geq 0\}$ and $\xi^{(k)} \in \text{Int}_{N_k} T_k$. And if $\xi^{(k)} = 0$, then V_k is the whole X . Note that V_k is a convex set in any case, i.e., $V_k^{\circ\circ} = V_k$. Then such a V satisfies the desired properties. Indeed it is easy to see $V_k^\circ \setminus \{0\} \subset \gamma_k$.

We will show that

$$V^\circ \supseteq \bigcap_k (Z_{k,\Gamma} \cup Z_{k,\epsilon}) \supseteq Z.$$

Here we emphasize that the inequality appearing in $Z_{k,\epsilon}$

$$(2.10) \quad \epsilon |x^{(k)}| \leq \sum_{j \in J_{\prec k}} |x^{(j)}|$$

and that in V_k for k with $\xi^{(k)} \neq 0$

$$(2.11) \quad \delta |\langle x^{(k)}, \xi^{(k)} \rangle| \geq \sum_{j \in J_{\succ k}} |x^{(j)}|$$

play an important role below.

Let $\sigma = \sigma_1 \sigma_2 \dots \sigma_\ell$ be an ℓ -length sequence where σ_k is either the symbols Γ or ϵ ($k = 1, 2, \dots, \ell$), and let us define

$$K_\sigma := Z_{1,\sigma_1} \cap Z_{2,\sigma_2} \cap \dots \cap Z_{\ell,\sigma_\ell}.$$

Then we have

$$\bigcap_k (Z_{k,\Gamma} \cup Z_{k,\epsilon}) = \bigcup_\sigma K_\sigma.$$

We now show that, for each sequence σ , we obtain $V^\circ \supset K_\sigma$ if we take T_k ($k = 1, 2, \dots, \ell$) and $\delta > 0$ sufficiently small.

Set

$$J_\Gamma(\sigma) := \{j \in \{1, 2, \dots, \ell\}; \sigma_j = \Gamma\}$$

and

$$J_\epsilon(\sigma) := \{j \in \{1, 2, \dots, \ell\}; \sigma_j = \epsilon\}.$$

Note that, for $k \in \{1, 2, \dots, \ell\}$ with $\hat{I}_k = I_k$, we have $k \in J_\Gamma(\sigma)$ and $k \notin J_\epsilon(\sigma)$, which implies, in particular, $J_\Gamma(\sigma)$ is non-empty. As both V_k and Γ_k are proper cones with its direction $\xi^{(k)}$ in N_k if $\xi^{(k)} \neq 0$, and as $\Gamma_k = \{0\}$ if $\xi^{(k)} = 0$, there exists a constant $M > 0$ such that

$$M |x^{(j)}| |y^{(j)}| \leq \langle x^{(j)}, y^{(j)} \rangle \leq |x^{(j)}| |y^{(j)}|$$

holds for $j \in J_\Gamma(\sigma)$ and $x = (x^{(1)}, \dots, x^{(\ell)}) \in K_\sigma$ and $y = (y^{(1)}, \dots, y^{(\ell)}) \in V$. Furthermore, by (2.10), there exists a constant $N > 0$ such that, for any $j \in J_\epsilon(\sigma)$, we have

$$|x^{(j)}| \leq \frac{N}{\epsilon} \sum_{\alpha \in J_\Gamma(\sigma) \cap J_{\prec j}} |x^{(\alpha)}|$$

for $x = (x^{(1)}, \dots, x^{(\ell)}) \in K_\sigma$. By noticing these facts, we obtain, for $x = (x^{(1)}, \dots, x^{(\ell)}) \in K_\sigma$ and $y = (y^{(1)}, \dots, y^{(\ell)}) \in V$.

$$\begin{aligned}
\langle x, y \rangle &= \sum_j \langle x^{(j)}, y^{(j)} \rangle = \sum_{j \in J_\Gamma(\sigma)} \langle x^{(j)}, y^{(j)} \rangle + \sum_{j \in J_\epsilon(\sigma)} \langle x^{(j)}, y^{(j)} \rangle \\
&\geq M \sum_{j \in J_\Gamma(\sigma)} |x^{(j)}| |y^{(j)}| - \sum_{j \in J_\epsilon(\sigma)} |x^{(j)}| |y^{(j)}| \\
&\geq M \sum_{j \in J_\Gamma(\sigma)} |x^{(j)}| |y^{(j)}| - \frac{N}{\epsilon^N} \sum_{j \in J_\epsilon(\sigma)} \left(\sum_{\alpha \in J_\Gamma(\sigma) \cap J_{\prec j}} |x^{(\alpha)}| \right) |y^{(j)}|.
\end{aligned}$$

Here, as $\Gamma_\alpha = \{0\}$ for α with $\xi^{(\alpha)} = 0$, we have $|x^{(\alpha)}| = 0$ for such an $\alpha \in J_\Gamma(\sigma)$ and the last term in the above inequalities is equal to

$$(2.12) \quad M \sum_{j \in J_\Gamma(\sigma)} |x^{(j)}| |y^{(j)}| - \frac{N}{\epsilon^N} \sum_{j \in J_\epsilon(\sigma)} \left(\sum_{\alpha \in J_\Gamma(\sigma) \cap J_{\prec j}, \xi^{(\alpha)} \neq 0} |x^{(\alpha)}| \right) |y^{(j)}|.$$

It follows from (2.11) that we get $\delta |y^{(\alpha)}| \geq \delta |\langle y^{(\alpha)}, \xi^{(\alpha)} \rangle| \geq |y^{(j)}|$ for α with $\xi^{(\alpha)} \neq 0$ and for $j \in J_{\succ \alpha}$ ($\Leftrightarrow \alpha \in J_{\prec j}$). Hence the (2.12) is lower bounded by

$$\begin{aligned}
&M \sum_{j \in J_\Gamma(\sigma)} |x^{(j)}| |y^{(j)}| - \frac{\delta N}{\epsilon^N} \sum_{j \in J_\epsilon(\sigma)} \sum_{\alpha \in J_\Gamma(\sigma) \cap J_{\prec j}, \xi^{(\alpha)} \neq 0} |x^{(\alpha)}| |y^{(\alpha)}| \\
&\geq M \sum_{j \in J_\Gamma(\sigma)} |x^{(j)}| |y^{(j)}| - \frac{\delta N \# J_\epsilon(\sigma)}{\epsilon^N} \sum_{\alpha \in J_\Gamma(\sigma)} |x^{(\alpha)}| |y^{(\alpha)}| \\
&= \left(M - \frac{\delta N \# J_\epsilon(\sigma)}{\epsilon^N} \right) \sum_{\alpha \in J_\Gamma(\sigma)} |x^{(\alpha)}| |y^{(\alpha)}|.
\end{aligned}$$

Note that the set $J_\Gamma(\sigma)$ is non-empty as noted above. Hence, if we take δ sufficiently small, $\langle x, y \rangle$ takes non-negative values for $x \in K_\sigma$ and $y \in V$, which implies $K_\sigma \subset V^\circ$. Hence we have shown the existence of G described in the theorem with $Z \subset G$.

Now we show that G described in the theorem satisfies $C_{M_k}(G)_q \subset p_k^\# \cup \{0\}$ for any k . We may assume $G = V^\circ$ where V was defined in the first part of the proof. Suppose that there exists a non-zero vector $\eta \in (T_{M_k} X)_q =$

$N' \times N_k$ such that

$$0 \neq \eta \in C_{M_k}(V^\circ)_q \cap \tau_k(p_k^{\circ a}).$$

Note that $\eta \in \tau_k(p_k^{\circ a})$ implies the existence of $0 \neq \eta^{(k)} \in N_k$ such that $\eta = (0_{N'}, \eta^{(k)})$ and $\langle \eta^{(k)}, \xi^{(k)} \rangle \leq 0$. Set, for any $\epsilon > 0$,

$$Q_\epsilon := \left\{ x = (x', x^{(k)}, x'') \in X; \begin{array}{l} |x'| < \epsilon \langle \eta^{(k)}, x^{(k)} \rangle, |x''| < \epsilon \\ |x^{(k)}| < \epsilon, x^{(k)} \in Q_\epsilon^{(k)} \end{array} \right\},$$

where $\{Q_\epsilon^{(k)}\}_{\epsilon > 0}$ is a family of open cone neighborhoods of the direction $\eta^{(k)}$ in N_k . Then $\eta \in C_{M_k}(V^\circ)_q$ means $Q_\epsilon \cap V^\circ \neq \emptyset$ for any $\epsilon > 0$. By noticing $\langle \eta^{(k)}, \xi^{(k)} \rangle \leq 0$, it follows from the definition of V that there exists a vector $v = (v', v^{(k)}, 0_{N''}) \in V$ such that we can find a positive constant $C > 0$ with

$$\langle x^{(k)}, v^{(k)} \rangle < -C|x^{(k)}| \quad (x^{(k)} \in Q_\epsilon^{(k)})$$

for any sufficiently small $\epsilon > 0$. Hence we have, for $x = (x', x^{(k)}, x'') \in Q_\epsilon$,

$$\langle x, v \rangle = \langle x', v' \rangle + \langle x^{(k)}, v^{(k)} \rangle \leq (\epsilon |\eta^{(k)}| |v'| - C) |x^{(k)}|.$$

As a result, if we take a sufficiently small $\epsilon > 0$, we have $\langle x, v \rangle < 0$ for any $x \in Q_\epsilon$, and thus, we get $Q_\epsilon \cap V^\circ = \emptyset$ which contradicts $Q_\epsilon \cap V^\circ \neq \emptyset$. Therefore we have obtained the conclusion. This completes the proof. \square

Remark 2.12. In the case $\ell = 2$ with $M_1 \subset M_2 \subset X$ we obtain the stalk formula computed in [15].

Now let us consider the mixed cases between specialization and microlocalization. We shall need some notations. Given a subset $K = \{i_1, \dots, i_k\} \subseteq \{1, \dots, \ell\}$, let $\chi_K = \{M_i, i \in K\}$, set $S_{i_j} = T_{M_{i_j}} \iota(M_{i_j}) \times_{M_{i_j}} M$ ($j = 1, \dots, k$) and $S_K = T_{M_{i_1}} \iota(M_{i_1}) \times_X \dots \times_X T_{M_{i_k}} \iota(M_{i_k})$ and let S_K^* be its dual. Given $C_{i_j} \subseteq S_{i_j}$, $j = 1, \dots, k$, we set for short $C_K := C_{i_1} \times_X \dots \times_X C_{i_k} \subset S_K$. Define \wedge_K as the composition of the Fourier-Sato transforms \wedge_{i_k} on S_{i_k} for each $i_k \in K$.

Let $I, J \subseteq \{1, \dots, \ell\}$ be such that $I \sqcup J = \{1, \dots, \ell\}$.

Lemma 2.13. *Let $V = V_I \times_X V_J$ be a multi-conic open subanalytic subset in $S_I \times_X S_J^*$ such that $V \cap \pi^{-1}(q)$ is convex for $q \in \pi(V)$. Then $H^k(V; \nu_{\chi_I}^{sa} \mu_{\chi_J}^{sa} F) \simeq \varinjlim_{Z, U} H_Z^k(U; F)$, where U ranges through the family of open subanalytic subsets in X with $C_{\chi_I}(X \setminus U) \cap \bigcup_{i \in I} \pi_i^{-1}(V_i) = \emptyset$ and $Z = \bigcap_{j \in J} Z_j$ with $C_{M_j}(Z_j) \subset (T_{M_j} X \setminus \tau_j(\text{Int}(V_j^{\circ a})))$. Here $(\cdot)^a$ denotes the inverse image of the antipodal map.*

Proof. We write \times instead of \times_X for short. Since V is convex the Fourier-Sato transform gives $H^k(V; \nu_{\chi_I}^{sa} \mu_{\chi_J}^{sa} F) \simeq H_{V_I \times V_J^\circ}^k(S; \nu_X^{sa} F)$. Consider the distinguished triangle

$$(2.13) \quad R\Gamma_{(S_I \setminus V_I) \times V_J^\circ} \nu_X^{sa} F \rightarrow R\Gamma_{S_I \times V_J^\circ} \nu_X^{sa} F \rightarrow R\Gamma_{V_I \times V_J^\circ} \nu_X^{sa} F \xrightarrow{+}$$

By Lemma 2.7 we have $H_{S_I \times V_J^\circ}^k(S; \nu_X^{sa} F) \simeq \varinjlim_{Z, U} H_Z^k(U; F)$, where U ranges

through the family of open subanalytic subsets of X such that $U \cap M = \pi(V)$ and $Z = \bigcap_{j \in J} Z_j$ with $C_{M_j}(Z_j) \subset (T_{M_j} X \setminus \tau_j(\text{Int}(V_j^{\circ a})))$. By Lemma 2.4 we also have $H_{(S_I \setminus V_I) \times V_J^\circ}^k(S; \nu_X^{sa} F) \simeq \varinjlim_{Z, U} H_Z^k(U; F)$, where U ranges

through the family of open subanalytic subsets of X such that $U \cap M = \pi(V)$ and $Z = \bigcap_{j=1}^\ell Z_j$ with $C_{M_j}(Z_j) \subset (T_{M_j} X \setminus \tau_j(\text{Int}(V_j^{\circ a})))$ if $j \in J$ and $C_{M_j}(Z_j) \subset (T_{M_j} X \setminus \tau_j(V_j))$ if $j \in I$. Thanks to the long exact sequence associated to (2.13) we obtain $H_{V_I \times V_J^\circ}^k(S; \nu_X^{sa} F) \simeq \varinjlim_{Z, U} H_Z^k(U \cap W; F)$, where

U ranges through the family of open subanalytic subsets of X such that $U \cap M = \pi(V)$, $Z = \bigcap_{j \in J} Z_j$ with $C_{M_j}(Z_j) \subset (T_{M_j} X \setminus \tau_j(\text{Int}(V_j^{\circ a})))$ and $W = \bigcup_{i \in I} (X \setminus Z_i)$ such that $C_{M_i}(Z_i) \subset (T_{M_i} X \setminus \tau_i(V_i))$. Then the result follows since, as in Lemma 2.4

$$C_{M_i}(Z_i) \subset (T_{M_i} X \setminus \tau_i(V_i)) \Leftrightarrow C_X(Z_i) \cap \pi_i^{-1}(V_i) = \emptyset.$$

□

Let $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)}) = (q; \xi_I, \xi_J) \in S_I \times_X S_J^*$. Locally we may identify S_J with its dual. Set for short $\nu_{\chi_I} \mu_{\chi_J} := \rho^{-1} \nu_{\chi_I}^{sa} \mu_{\chi_J}^{sa}$. As a consequence of Lemma 2.13 we have

Corollary 2.14. *Let $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)}) \in S_I \times_X S_J^*$, and let $F \in D^b(k_{X_{sa}})$. Then $H^k(\nu_{\chi_I} \mu_{\chi_J} F)_p \simeq \varinjlim_{Z, W_\epsilon} H_Z^k(W_\epsilon; F)$, where $W_\epsilon = W \cap B_\epsilon$, with $W \in \text{Cone}_\chi(q; \xi_I, 0_J)$, B_ϵ is an open ball containing q of radius $\epsilon > 0$ and Z runs through a family of closed sets in the form $Z_1 \cap Z_2 \cap \cdots \cap Z_\ell$ with each Z_k being closed subanalytic in X and $C_{M_k}(Z_k)_q \subset p_k^\# \cup \{0\}$ ($k = 1, 2, \dots, \ell$).*

Proof. The result follows since for any subanalytic conic neighborhood V of $(q; \xi_I, 0_J)$, any $U \in \text{Op}(X_{sa})$ such that $C_\chi(X \setminus U) \cap V = \emptyset$ contains $W \cap B_\epsilon$, $q \in B_\epsilon$, $\epsilon > 0$, $W \in \text{Cone}(q; \xi_I, 0_J)$. Moreover, by definition of multi-cone we may assume $W = \bigcap_{j=1}^\ell W_j$ such that $C_{M_j}(\overline{W_j})_q \subset p_j^\# \cup \{0\}$ if $j \in I$ and $W_j = X$ if $j \in J$. \square

As in Theorem 2.11 we can find a cofinal family to the family of closed subsets defining the stalk formula in Corollary 2.14 which (locally) consists of convex cones and we can formulate the stalk formula in the mixed case.

Theorem 2.15. *Let $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)}) \in S_I \times_X S_J^*$, and let $F \in D^b(k_{X_{sa}})$. Then we have*

$$(2.14) \quad H^k(\nu_{\chi_I} \mu_{\chi_J} F)_p \simeq \varinjlim_{G, W_\epsilon} H_G^k(W_\epsilon; F).$$

Here $W_\epsilon = W \cap B_\epsilon$, with $W \in \text{Cone}_\chi(q; \xi_I, 0_J)$, B_ϵ is an open ball of radius $\epsilon > 0$ containing q and a closed subanalytic subset $G = M \times \left(\sum_{k=1}^\ell G_k \right)$ with G_k being a closed subanalytic convex cone in N satisfying $G_k \setminus \{0\} \subset \gamma_k$, where γ_k is defined in (2.8).

3 Multi-microlocalization and microsupport

In this section we give an estimate of the microsupport of multi-microlocalization. The main point is to find a suitable ambient space: this is done (via Hamiltonian isomorphism) by identifying T^*S_χ with the normal deformation of T^*X with respect to a suitable family of submanifolds χ^* .

3.1 Geometry

Let X be a real analytic manifold and consider a family of submanifolds $\chi = \{M_1, \dots, M_\ell\}$ satisfying H1, H2 and H3. We consider the conormal bundle T^*X with local coordinates $(x^{(0)}, x^{(1)}, \dots, x^{(\ell)}; \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(\ell)})$, where $x^{(j)} = (x_{j_1}, \dots, x_{j_p})$ with $\hat{I}_j = \{j_1, \dots, j_p\}$ etc. We use the notations in § 2.1; for example, we set $S_i := T_{M_i} \iota(M_i) \times_X M$. Let $I, J \subseteq \{1, \dots, \ell\}$ be such that $I \sqcup J = \{1, \dots, \ell\}$. Recall that

$$\begin{aligned} S_\chi &= S_1 \times_X \cdots \times_X S_\ell, \\ S_\chi^* &= S_1^* \times_X \cdots \times_X S_\ell^*, \\ S_I \times_M S_J^* &= \left(\times_{M, i \in I} S_i \right) \times \left(\times_{M, j \in J} S_j^* \right). \end{aligned}$$

Then we consider a mapping

$$\begin{aligned} H_{IJ}: T^*S_\chi &\ni (x^{(0)}, x^{(1)}, \dots, x^{(\ell)}; \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(\ell)}) \\ &\mapsto (x^{(0)}, (x^{(i)})_{i \in I}, (\xi^{(j)})_{j \in J}; \eta^{(0)}, (\xi^{(i)})_{i \in I}, (-x^{(j)})_{j \in J}) \in T^*(S_I \times_M S_J^*). \end{aligned}$$

Note that H_{IJ} is induced by the Hamiltonian isomorphisms $T^*S_J \xrightarrow{\sim} T^*S_J^*$.

Proposition 3.1. *H_{IJ} gives a bundle isomorphism over M ; that is, H_{IJ} does not depend on the choice of local coordinates.*

Proof. Let $\varphi: X \rightarrow X$ be a local coordinate transformation near any $x \in X$. We may assume that $X = \mathbb{R}^n$ with coordinates $x = (x^{(0)}, x^{(1)}, \dots, x^{(\ell)})$, where M is given by $(x^{(0)}, 0, \dots, 0)$, and φ is given by

$$y^{(j)} = \varphi^{(j)}(x^{(0)}, x^{(1)}, \dots, x^{(\ell)}) \quad (j = 0, 1, \dots, \ell).$$

Here $\varphi^{(j)}(x) = (\varphi_{j_1}(x), \dots, \varphi_{j_{p(j)}}(x))$ with $\hat{I}_j = \{j_1, \dots, j_{p(j)}\}$. This induces a coordinate transformation

$$T^*X \ni (x; \xi) \mapsto (y; \eta) \in T^*X$$

defined by

$$\begin{cases} y^{(j)} = \varphi^{(j)}(x), \\ \xi^{(j)} = \sum_{i=0}^{\ell} \text{tr} \left[\frac{\partial \varphi^{(i)}}{\partial x^{(j)}}(x) \right] \eta^{(i)}, \end{cases}$$

where

$$\frac{\partial \varphi^{(i)}}{\partial x^{(j)}}(x) = \begin{bmatrix} \frac{\partial \varphi_{i_1}}{\partial x_{j_1}}(x) & \cdots & \frac{\partial \varphi_{i_1}}{\partial x_{j_{p(j)}}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_{i_{p(i)}}}{\partial x_{j_1}}(x) & \cdots & \frac{\partial \varphi_{i_{p(i)}}}{\partial x_{j_{p(j)}}}(x) \end{bmatrix}$$

is a $p(i) \times p(j)$ -matrix, and t means the transpose of a matrix. Set $J_j(x^{(0)}) := \frac{\partial \varphi^{(j)}}{\partial x^{(j)}}(x^{(0)}, 0)$ for short. Then the coordinate transformation

$$(x^{(0)}, x^{(1)}, \dots, x^{(\ell)}) \mapsto (y^{(0)}, y^{(1)}, \dots, y^{(\ell)})$$

on S_χ is given by

$$(3.1) \quad \begin{cases} y^{(0)} = \varphi^{(0)}(x^{(0)}, 0), \\ y^{(j)} = J_j(x^{(0)}) x^{(j)} \quad (j = 1, \dots, \ell). \end{cases}$$

The Jacobian matrix of (3.1) is

$$\begin{bmatrix} J_0(x^{(0)}) & 0 & \cdots & 0 \\ \frac{\partial J_1}{\partial x^{(0)}}(x^{(0)}) x^{(1)} & J_1(x^{(0)}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial J_\ell}{\partial x^{(0)}}(x^{(0)}) x^{(\ell)} & 0 & \cdots & J_\ell(x^{(0)}) \end{bmatrix}.$$

Thus the the coordinate transformation

$$\begin{aligned} (x^{(0)}, x^{(1)}, \dots, x^{(\ell)}; \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(\ell)}) \\ \mapsto (y^{(0)}, y^{(1)}, \dots, y^{(\ell)}; \eta^{(0)}, \eta^{(1)}, \dots, \eta^{(\ell)}) \end{aligned}$$

on T^*S_χ is given by (3.1) and

$$(3.2) \quad \begin{cases} \xi^{(0)} = {}^t J_0(x^{(0)}) \eta^{(0)} + \sum_{i=1}^{\ell} {}^t x^{(i)} \frac{\partial {}^t J_i}{\partial x^{(0)}}(x^{(0)}) \eta^{(i)}, \\ \xi^{(j)} = {}^t J_j(x^{(0)}) \eta^{(j)} \quad (j = 1, \dots, \ell). \end{cases}$$

Next, consider the the coordinate transformation on $S_I \times_M S_J^*$. After a permutation, we may assume that $I = \{1, \dots, p\}$, $J = \{p+1, \dots, \ell\}$. Then the coordinate transformation

$$(x^{(0)}, (x^{(i)})_{i=1}^p, (\xi^{(j)})_{j=p+1}^\ell) \mapsto (y^{(0)}, (y^{(i)})_{i=1}^p, (\eta^{(j)})_{j=p+1}^\ell)$$

on $S_I \times_M S_J^*$ is given by

$$(3.3) \quad \begin{cases} y^{(0)} = \varphi^{(0)}(x^{(0)}, 0), \\ y^{(i)} = J_i(x^{(0)}) x^{(i)} \quad (i = 1, \dots, p), \\ \eta^{(j)} = {}^t J_j^{-1}(x^{(0)}) \xi^{(j)} \quad (j = p+1, \dots, \ell). \end{cases}$$

The Jacobian matrix of (3.3) is

$$\begin{bmatrix} J_0(x^{(0)}) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \frac{\partial J_1}{\partial x^{(0)}}(x^{(0)}) x^{(1)} & J_1(x^{(0)}) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \frac{\partial J_p}{\partial x^{(0)}}(x^{(0)}) x^{(p)} & 0 & \cdots & J_p(x^{(0)}) & 0 & \cdots & 0 \\ \frac{\partial {}^t J_{p+1}^{-1}}{\partial x^{(0)}}(x^{(0)}) \xi^{(p+1)} & 0 & \cdots & 0 & {}^t J_{p+1}^{-1}(x^{(0)}) & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \ddots & \vdots \\ \frac{\partial {}^t J_\ell^{-1}}{\partial x^{(0)}}(x^{(0)}) \xi^{(\ell)} & 0 & \cdots & 0 & 0 & \cdots & {}^t J_\ell^{-1}(x^{(0)}) \end{bmatrix}.$$

Thus the the coordinate transformation

$$\begin{aligned} & (x^{(0)}, (x^{(i)})_{i=1}^p, (\xi^{(j)})_{j=p+1}^\ell; \xi^{(0)}, (\xi^{(i)})_{i=1}^p, (-x^{(j)})_{j=p+1}^\ell) \\ & \mapsto (y^{(0)}, (y^{(i)})_{i=1}^p, (\eta^{(j)})_{j=p+1}^\ell; \eta^{(0)}, (\eta^{(i)})_{i=1}^p, (-y^{(j)})_{j=p+1}^\ell) \end{aligned}$$

on $T^*S_\chi^*$ is given by (3.3) and

$$\begin{cases} \xi^{(0)} = {}^t J_0(x^{(0)}) \eta^{(0)} + \sum_{i=1}^p {}^t x^{(i)} \frac{\partial {}^t J_i}{\partial x^{(0)}}(x^{(0)}) \eta^{(i)} - \sum_{j=p+1}^\ell {}^t \xi^{(j)} \frac{\partial J_j^{-1}}{\partial x^{(0)}}(x^{(0)}) y^{(j)}, \\ \xi^{(i)} = {}^t J_i(x^{(0)}) \eta^{(i)} \quad (i = 1, \dots, p), \\ x^{(j)} = J_j^{-1}(x^{(0)}) y^{(j)} \quad (j = p+1, \dots, \ell). \end{cases}$$

Since

$$\frac{\partial J_j^{-1}}{\partial x^{(0)}}(x^{(0)}) = -J_j^{-1}(x^{(0)}) \frac{\partial J_j}{\partial x^{(0)}}(x^{(0)}) J_j^{-1}(x^{(0)}),$$

we have

$$\begin{aligned} -({}^t\xi^{(j)} \frac{\partial J_j^{-1}}{\partial x^{(0)}}(x^{(0)}) y^{(j)})_k &= ({}^t\xi^{(j)} J_j^{-1}(x^{(0)}) \frac{\partial J_j}{\partial x^{(0)}}(x^{(0)}) J_j^{-1}(x^{(0)}) y^{(j)})_k \\ &= ({}^t\eta^{(j)} \frac{\partial J_j}{\partial x^{(0)}}(x^{(0)}) x^{(j)})_k = \sum_{\mu, \nu \in \hat{I}_j} \eta_\mu^{(j)} \left(\frac{\partial J_j}{\partial x_k^{(0)}}(x^{(0)}) \right)_{\mu, \nu} x_\nu^{(j)} \\ &= ({}^tx^{(j)} \frac{\partial {}^tJ_j}{\partial x^{(0)}}(x^{(0)}) \eta^{(j)})_k. \end{aligned}$$

Therefore

$$-\sum_{j=p+1}^{\ell} {}^t\xi^{(j)} \frac{\partial J_p^{-1}}{\partial x^{(0)}}(x^{(0)}) y^{(j)} = \sum_{j=p+1}^{\ell} {}^tx^{(j)} \frac{\partial {}^tJ_j}{\partial x^{(0)}}(x^{(0)}) \eta^{(j)}.$$

Thus we can prove that

$$(3.4) \quad H_{IJ}: T^*S_\chi \xrightarrow{\sim} T^*(S_I \times_M S_J^*). \quad \square$$

Hence, using Proposition 5.5.5 of [7] repeatedly, we obtain:

Proposition 3.2. *Let $I, J \subseteq \{1, \dots, \ell\}$ be such that $I \sqcup J = \{1, \dots, \ell\}$. Then, under the identification by (3.4), for any $F \in D^b(k_X)$ it follows that*

$$\begin{aligned} T^*S_\chi &\stackrel{=}{=} T^*(S_I \times_M S_J^*) \\ \cup &\quad \cup \\ \text{SS}(\nu_\chi(F)) &= \text{SS}(\nu_{\chi_I} \mu_{\chi_J}(F)). \end{aligned}$$

In particular, it follows that

$$\begin{aligned} T^*S_\chi &\stackrel{=}{=} T^*S_\chi^* \\ \cup &\quad \cup \\ \text{SS}(\nu_\chi(F)) &= \text{SS}(\mu_\chi(F)). \end{aligned}$$

Next, we study the relation between the normal deformations of T^*X with respect to $\chi^* := \{T_{M_1}^*X, \dots, T_{M_\ell}^*X\}$ and of X with respect to χ . We

denote by $\widetilde{T^*X}_{\chi^*} := \widetilde{T^*X}_{T_{M_1}^*X, \dots, T_{M_\ell}^*X}$ the normal deformation of T^*X with respect to χ^* and by S_{χ^*} its zero-section. Set $x := (x^{(0)}, x^{(1)}, \dots, x^{(\ell)})$, $\xi := (\xi^{(0)}, \xi^{(1)}, \dots, \xi^{(\ell)})$ and $t := (t_1, \dots, t_\ell)$. We have a mapping

$$\widetilde{T^*X}_{\chi^*} \ni (x; \xi; t) \mapsto (\mu_x(x; t); \mu_\xi(\xi; t)) \in T^*X$$

defined by

$$\begin{aligned} \mu_x(x; t) &:= (t_{\hat{j}_0} x^{(0)}, t_{\hat{j}_1} x^{(1)}, \dots, t_{\hat{j}_\ell} x^{(\ell)}), \\ \mu_\xi(\xi; t) &:= (t_{\hat{j}_0^c} \xi^{(0)}, t_{\hat{j}_1^c} \xi^{(1)}, \dots, t_{\hat{j}_\ell^c} \xi^{(\ell)}), \end{aligned}$$

where $\hat{j}_j^c := \{1, \dots, \ell\} \setminus \hat{j}_j$ ($j = 0, 1, \dots, \ell$). In particular $\hat{j}_0^c = \{1, \dots, \ell\}$ since $\hat{j}_0 = \emptyset$. In particular $t_{\hat{j}_0} = 1$ and $t_{\hat{j}_0^c} = t_1 \cdots t_\ell$.

Theorem 3.3. *As vector bundles, there exist the following canonical isomorphism:*

$$S_{\chi^*} \simeq T^*S_\chi \simeq T^*S_\chi^*.$$

Proof. Let $\varphi: X \rightarrow X$ be a local coordinate transformation near any $x \in X$, and retain the notation of the proof of Proposition 3.1. The coordinate transformation $(x; \xi; t) \mapsto (y; \eta; t)$ on $\widetilde{T^*X}_{\chi^*} \setminus S_{\chi^*}$ is given by

$$\begin{cases} y^{(j)} = \frac{1}{t_{\hat{j}_j}} \varphi^{(j)}(t_{\hat{j}} x), \\ \xi^{(j)} = \sum_{i=0}^{\ell} t_{\hat{j}_i} \left[\frac{\partial \varphi^{(i)}}{\partial x^{(j)}}(t_{\hat{j}} x) \right] \frac{t_{\hat{j}_i^c}}{t_{\hat{j}_j^c}} \eta^{(i)}, \end{cases}$$

where $t_{\hat{j}} x := \mu_x(x; t) = (t_{\hat{j}_0} x^{(0)}, t_{\hat{j}_1} x^{(1)}, \dots, t_{\hat{j}_\ell} x^{(\ell)})$. Let us consider the coordinate transformation on S_{χ^*} . We write for short $t \rightarrow 0$ instead of

$(t_1, \dots, t_\ell) \rightarrow (0, \dots, 0)$. Set $J_j(x^{(0)}) := \frac{\partial \varphi^{(j)}}{\partial x^{(j)}}(x^{(0)}, 0)$ for short. Then, by Proposition 1.5 of [4] on S_{χ^*}

$$\begin{cases} y^{(0)} = \varphi^{(0)}(x^{(0)}, 0), \\ y^{(j)} = J_j(x^{(0)}) x^{(j)}, \quad (j = 1, \dots, \ell), \end{cases}$$

that means $y_k = \sum_{p \in \hat{I}_k} \frac{\partial \varphi_k}{\partial x_p}(x^{(0)}, 0) x_p$ for all $k \in \hat{I}_j$. Concerning the variable $\xi^{(0)}$, as in the proof of Proposition 3.1 we get

$$\xi^{(0)} = {}^t[\frac{\partial \varphi^{(0)}}{\partial x^{(0)}}(t_j x)] \eta^{(0)} + \sum_{i=1}^{\ell} {}^t[\frac{\partial \varphi^{(i)}}{\partial x^{(0)}}(t_j x)] \frac{t_{j_i^c}}{t_{j_0^c}} \eta^{(i)}.$$

Let $M_i = \{x_k = 0; k \in I_i\}$ and $I_i = \hat{I}_{j_1} \sqcup \dots \sqcup \hat{I}_{j_p}$. By expanding ${}^t[\frac{\partial \varphi^{(i)}}{\partial x^{(0)}}(t_j x)]$ along the submanifold M_i , we obtain

$$\begin{aligned} {}^t[\frac{\partial \varphi^{(i)}}{\partial x^{(0)}}(t_j x)] \frac{t_{j_i^c}}{t_{j_0^c}} \eta^{(i)} &= {}^t[\frac{\partial \varphi^{(i)}}{\partial x^{(0)}}(t_j x)] \Big|_{M_i} \frac{t_{j_i^c}}{t_{j_0^c}} \eta^{(i)} \\ &+ \sum_{k \in I_i} x_k {}^t[\frac{\partial^2 \varphi^{(i)}}{\partial x_k \partial x^{(0)}}(t_j x)] \Big|_{M_i} \frac{t_{j_i^c} t_{j_k}}{t_{j_0^c}} \eta^{(i)} + \dots \end{aligned}$$

Since $\varphi^{(i)}(t_j x)|_{M_i} = 0$ and $\hat{I}_0 \cap I_i = \emptyset$ we have $\frac{\partial \varphi^{(i)}}{\partial x^{(0)}}(t_j x) \Big|_{M_i} = 0$. Moreover

$$\begin{cases} \frac{t_{j_i^c} t_{j_i}}{t_{j_0^c}} = 1, \\ \frac{t_{j_i^c} t_{j_k}}{t_{j_0^c}} \rightarrow 0, \quad (k \neq i), \end{cases}$$

when $t \rightarrow 0$. This is because $\hat{J}_k \subsetneq \hat{J}_i$ when $k \in \{j_1, \dots, j_p\}$, $k \neq i$ by Lemma 1.1 and (1.7). In a similar way the higher order terms vanish when $t \rightarrow 0$. Hence on S_{χ^*}

$$\begin{aligned} \xi^{(0)} &= {}^t[\frac{\partial \varphi^{(0)}}{\partial x^{(0)}}(x^{(0)}, 0)] \eta^{(0)} + \sum_{i=1}^{\ell} {}^t x^{(i)} {}^t[\frac{\partial^2 \varphi^{(i)}}{\partial x^{(i)} \partial x^{(0)}}(x^{(0)}, 0)] \eta^{(i)} \\ &= {}^t J_0(x^{(0)}) \eta^{(0)} + \sum_{i=1}^{\ell} {}^t x^{(i)} \frac{\partial {}^t J_i}{\partial x^{(0)}}(x^{(0)}) \eta^{(i)}. \end{aligned}$$

Concerning the variable $\xi^{(j)}$ ($j \neq 0$), we get

$$\xi^{(j)} = \sum_{i=0}^{\ell} {}^t[\frac{\partial \varphi^{(i)}}{\partial x^{(j)}}(t_j x)] \frac{t_{j_i^c}}{t_{j_j^c}} \eta^{(i)}.$$

(i) If $\hat{J}_j^c \subsetneq \hat{J}_i^c \Leftrightarrow \hat{J}_i \subsetneq \hat{J}_j$ we have $\frac{t_{\hat{J}_i^c}}{t_{\hat{J}_j^c}} \rightarrow 0$ ($t \rightarrow 0$).

(ii) If $\hat{J}_i \supsetneq \hat{J}_j$ or $\hat{J}_i \cap \hat{J}_j = \emptyset$, we have $\hat{I}_j \cap I_i = \emptyset$.

By expanding $\frac{\partial \varphi^{(i)}}{\partial x^{(j)}}(t_j x)$ along the submanifold M_i , we obtain

$$\begin{aligned} {}^t[\frac{\partial \varphi^{(i)}}{\partial x^{(j)}}(t_j x)] \frac{t_{\hat{J}_i^c}}{t_{\hat{J}_j^c}} \eta^{(i)} &= {}^t[\frac{\partial \varphi^{(i)}}{\partial x^{(j)}}(t_j x)] \Big|_{M_i} \frac{t_{\hat{J}_i^c}}{t_{\hat{J}_j^c}} \eta^{(i)} \\ &+ \sum_{k \in I_i} x_k {}^t[\frac{\partial^2 \varphi^{(i)}}{\partial x_k \partial x^{(j)}}(t_j x)] \Big|_{M_i} \frac{t_{\hat{J}_i^c} t_{\hat{J}_k}}{t_{\hat{J}_j^c}} \eta^{(i)} + \dots \end{aligned}$$

Since $\varphi^{(i)}(t_j x)|_{M_i} = 0$ and $\hat{I}_j \cap I_i = \emptyset$ we have $\frac{\partial \varphi^{(i)}}{\partial x^{(j)}}(t_j x) \Big|_{M_i} = 0$. Moreover

$$\frac{t_{\hat{J}_i^c} t_{\hat{J}_k}}{t_{\hat{J}_j^c}} \rightarrow 0$$

when $t \rightarrow 0$ since $\hat{I}_k \subseteq I_i \Rightarrow \hat{J}_k \supsetneq \hat{J}_i$ by Lemma 1.1 and $\hat{J}_i^c \cup \hat{J}_k = \{1, \dots, \ell\} \supsetneq \hat{J}_j^c$ when $j \neq 0$. Hence on S_{χ^*}

$$\xi^{(j)} = {}^t[\frac{\partial \varphi^{(j)}}{\partial x^{(j)}}(x^{(0)}, 0)] \eta^{(j)} = {}^t J_j(x^{(0)}) \eta^{(j)}.$$

Summarizing, the coordinate transformation on S_{χ^*} is given by

$$\begin{cases} y^{(0)} = \varphi^{(0)}(x^{(0)}, 0), \\ y^{(i)} = J_i(x^{(0)}) x^{(i)} \quad (1 \leq i \leq \ell), \\ \xi^{(0)} = {}^t J_0(x^{(0)}) \eta^{(0)} + \sum_{i=1}^{\ell} {}^t x^{(i)} \frac{\partial {}^t J_i}{\partial x^{(0)}}(x^{(0)}) \eta^{(i)}, \\ \xi^{(i)} = {}^t J_i(x^{(0)}) \eta^{(i)} \quad (1 \leq i \leq \ell). \end{cases}$$

This is nothing but (3.1), (3.2). \square

Example 3.4. Let $X = \mathbb{C}^2$ with coordinates (z_1, z_2) and consider T^*X with coordinates $(z; \eta) = (z_1, z_2; \eta_1, \eta_2)$. Set $t = (t_1, t_2) \in (\mathbb{R}^+)^2$.

1. (Majima) Let $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then $\chi^* = \{T_{M_1}^* X, T_{M_2}^* X\}$ and we have a map

$$\begin{aligned}\widetilde{T^* X} &\rightarrow T^* X, \\ (z; \eta; t) &\mapsto (\mu_z(z; t); \mu_\eta(\eta; t)),\end{aligned}$$

which is defined by

$$\begin{aligned}\mu_z(z; t) &= (t_1 z_1, t_2 z_2), \\ \mu_\eta(\eta; t) &= (t_2 \eta_1, t_1 \eta_2).\end{aligned}$$

By Theorem 3.3 we have $S_\chi^* \simeq T^*(T_{M_1} X \times_X T_{M_2} X) \simeq T^*(T_{M_1}^* X \times_X T_{M_2}^* X)$.

2. (Takeuchi) Let $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. Then $\chi^* = \{T_{M_1}^* X, T_{M_2}^* X\}$ and we have a map

$$\begin{aligned}\widetilde{T^* X} &\rightarrow T^* X, \\ (z; \eta; t) &\mapsto (\mu_z(z; t); \mu_\eta(\eta; t)),\end{aligned}$$

which is defined by

$$\begin{aligned}\mu_z(z; t) &= (t_1 z_1, t_1 t_2 z_2), \\ \mu_\eta(\eta; t) &= (t_2 \eta_1, \eta_2).\end{aligned}$$

By Theorem 3.3 we have $S_\chi^* \simeq T^*(T_{M_1} M_2 \times_X T_{M_2} X) \simeq T^*(T_{M_1}^* M_2 \times_X T_{M_2}^* X)$.

Example 3.5. Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and consider $T^* X$ with coordinates $(x; \xi) = (x_1, x_2, x_3; \xi_1, \xi_2, \xi_3)$. Set $t = (t_1, t_2, t_3) \in (\mathbb{R}^+)^3$.

1. (Majima) Let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi^* = \{T_{M_1}^* X, T_{M_2}^* X, T_{M_3}^* X\}$ and we have a map

$$\begin{aligned}\widetilde{T^* X} &\rightarrow T^* X, \\ (x; \xi; t) &\mapsto (\mu_x(x; t); \mu_\xi(\xi; t)),\end{aligned}$$

which is defined by

$$\begin{aligned}\mu_x(x; t) &= (t_1 x_1, t_2 x_2, t_3 x_3), \\ \mu_\xi(\xi; t) &= (t_2 t_3 \xi_1, t_1 t_3 \xi_2, t_2 t_3 \xi_3).\end{aligned}$$

By Theorem 3.3 we have $S_\chi^* \simeq T^*(T_{M_1} X \times_X T_{M_2} X \times_X T_{M_2} X) \simeq T^*(T_{M_1}^* X \times_X T_{M_2}^* X \times_X T_{M_3}^* X)$.

2. (Takeuchi) Let $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi^* = \{T_{M_1}^* X, T_{M_2}^* X, T_{M_3}^* X\}$ and we have a map

$$\begin{aligned}\widetilde{T^* X} &\rightarrow T^* X, \\ (x; \xi; t) &\mapsto (\mu_x(x; t); \mu_\xi(\xi; t)),\end{aligned}$$

which is defined by

$$\begin{aligned}\mu_x(x; t) &= (t_1 x_1, t_1 t_2 x_2, t_1 t_2 t_3 x_3), \\ \mu_\xi(\xi; t) &= (t_2 t_3 \xi_1, t_3 \xi_2, \xi_3).\end{aligned}$$

By Theorem 3.3 we have $S_\chi^* \simeq T^*(T_{M_1} M_2 \times_X T_{M_2} M_3 \times_X T_{M_2} X) \simeq T^*(T_{M_1}^* M_2 \times_X T_{M_2}^* M_3 \times_X T_{M_3}^* X)$.

3. (Mixed) Let $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi^* = \{T_{M_1}^* X, T_{M_2}^* X, T_{M_3}^* X\}$ and we have a map

$$\begin{aligned}\widetilde{T^* X} &\rightarrow T^* X, \\ (x; \xi; t) &\mapsto (\mu_x(x; t); \mu_\xi(\xi; t)),\end{aligned}$$

which is defined by

$$\begin{aligned}\mu_x(x; t) &= (t_1 x_1, t_1 t_2 x_2, t_1 t_3 x_3), \\ \mu_\xi(\xi; t) &= (t_2 t_3 \xi_1, t_3 \xi_2, t_2 \xi_3).\end{aligned}$$

By Theorem 3.3 we have $S_\chi^* \simeq T^*(T_{M_1} (M_2 \cap M_3) \times_X T_{M_2} X \times_X T_{M_2} X) \simeq T^*(T_{M_1}^* (M_2 \cap M_3) \times_X T_{M_2}^* X \times_X T_{M_3}^* X)$.

3.2 Estimate of microsupport

In this section we shall prove an estimate for the microsupport of the multi-specialization and multi-microlocalization of a sheaf on X . We refer to [7] for the theory of microsupport of sheaves.

Theorem 3.6. *Let $F \in D^b(k_X)$. Then*

$$\mathrm{SS}(\nu_\chi(F)) = \mathrm{SS}(\mu_\chi(F)) \subseteq C_{\chi^*}(\mathrm{SS}(F)).$$

Since the problem is local, we may assume that $X = \mathbb{R}^n$ with coordinates (x_1, \dots, x_n) . Let I_k ($k = 1, 2, \dots, \ell$) in $\{1, 2, \dots, n\}$ such that each submanifold M_k is given by

$$\{x = (x_1, \dots, x_n) \in \mathbb{R}^n; x_i = 0 (i \in I_k)\}.$$

Then Theorem 3.6 follows from Lemma 1.4 and the following theorem:

Theorem 3.7. *Let $F \in D^b(k_X)$ and take a point*

$$p_0 = (x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(\ell)}; \xi_0^{(0)}, \xi_0^{(1)}, \dots, \xi_0^{(\ell)}) \in T^*S_\chi.$$

Assume that $p_0 \in \mathrm{SS}(\nu_\chi(F))$. Then there exist sequences

$$\begin{aligned} & \{(c_{1,k}, \dots, c_{\ell,k})\}_{k=1}^\infty \subset (\mathbb{R}^+)^{\ell}, \\ & \{(x_k^{(0)}, x_k^{(1)}, \dots, x_k^{(\ell)}; \xi_k^{(0)}, \xi_k^{(1)}, \dots, \xi_k^{(\ell)})\}_{k=1}^\infty \subset \mathrm{SS}(F), \end{aligned}$$

such that

$$\left\{ \begin{aligned} & \lim_{k \rightarrow \infty} c_{j,k} = \infty, \quad (j = 1, \dots, \ell), \\ & \lim_{k \rightarrow \infty} (x_k^{(0)}, x_k^{(1)} c_{\hat{J}_1, k}, \dots, x_k^{(\ell)} c_{\hat{J}_\ell, k}; \xi_k^{(0)} c_k, \xi_k^{(1)} c_{\hat{J}_1^c, k}, \dots, \xi_k^{(\ell)} c_{\hat{J}_\ell^c, k}) \\ & \quad = (x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(\ell)}; \xi_0^{(0)}, \xi_0^{(1)}, \dots, \xi_0^{(\ell)}), \end{aligned} \right.$$

where $c_k := \prod_{j=1}^{\ell} c_{j,k}$, $\hat{J}_j^c := \{1, \dots, \ell\} \setminus \hat{J}_j$, and $c_{J,k} := \prod_{j \in J} c_{j,k}$ for any $J \subseteq \{1, \dots, \ell\}$.

Proof. Let $(x; t) = (x^{(0)}, x^{(1)}, \dots, x^{(\ell)}; t_1, \dots, t_\ell)$ be the coordinates in \tilde{X} . It follows from the estimate of the microsupport of the inverse image of a closed embedding (Lemma 6.2.1 (ii) and Proposition 6.2.4 (iii) of [7]) that there exists a sequence

$$\{(x_k^{(0)}, x_k^{(1)}, \dots, x_k^{(\ell)}; t_{1,k}, \dots, t_{\ell,k}; \xi_k^{(0)}, \xi_k^{(1)}, \dots, \xi_k^{(\ell)}; \tau_{1,k}, \dots, \tau_{\ell,k})\}_{k=1}^\infty$$

in $\text{SS}(Rj_{\Omega*}\tilde{p}^{-1}F)$ such that for any $j = 1, \dots, \ell$

$$\begin{cases} \lim_{k \rightarrow \infty} x_k^{(j)} = x_0^{(j)}, & \lim_{k \rightarrow \infty} \xi_k^{(j)} = \xi_0^{(j)}, & \lim_{k \rightarrow \infty} t_{j,k} = 0, \\ \lim_{k \rightarrow \infty} |(t_{1,k}, \dots, t_{\ell,k})| \cdot |(\tau_{1,k}, \dots, \tau_{\ell,k})| = 0. \end{cases}$$

By Theorem 6.3.1 of [7] we have

$$\text{SS}(Rj_{\Omega*}\tilde{p}^{-1}F) \subseteq \text{SS}(\tilde{p}^{-1}F) \hat{+} N^*(\Omega).$$

By Proposition 5.4.5 of [7], we have

$$\begin{aligned} \text{SS}(\tilde{p}^{-1}F) &= \tilde{p}_d(\tilde{p}_\pi^{-1}(\text{SS}(F))) \\ &= \{(x^{(0)}, \frac{x^{(1)}}{t_{j_1}}, \dots, \frac{x^{(\ell)}}{t_{j_\ell}}; t_1, \dots, t_\ell; \xi^{(0)}, t_{j_1}\xi^{(1)}, \dots, t_{j_\ell}\xi^{(\ell)}; \tau_1, \dots, \tau_\ell); \\ &\quad t_j > 0 \ (j = 1, \dots, \ell), (x^{(0)}, x^{(1)}, \dots, x^{(\ell)}; \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(\ell)}) \in \text{SS}(F)\}, \end{aligned}$$

where we did not calculate the terms in the variables τ_j ($j = 1, \dots, \ell$) since we are not going to use them. Thanks to Remark 6.2.8 (ii) of [7] and the fact that $N^*(\Omega) \subset \{(x; t; 0; \tau)\}$, for each $k \in \mathbb{N}$ we get sequences

$$\begin{aligned} \{(x_{k,m}^{(0)}, x_{k,m}^{(1)}, \dots, x_{k,m}^{(\ell)}; \xi_{k,m}^{(0)}, \xi_{k,m}^{(1)}, \dots, \xi_{k,m}^{(\ell)})\}_{m=1}^\infty &\subset \text{SS}(F), \\ \{(t_{1,k,m}, \dots, t_{\ell,k,m})\} &\subset (\mathbb{R}^+)^{\ell}, \end{aligned}$$

such that

$$\begin{aligned} \lim_{m \rightarrow \infty} (x_{k,m}^{(0)}, \frac{x_{k,m}^{(1)}}{t_{j_1,k,m}}, \dots, \frac{x_{k,m}^{(\ell)}}{t_{j_\ell,k,m}}; t_{1,k,m}, \dots, t_{\ell,k,m}) \\ = (x_k^{(0)}, x_k^{(1)}, \dots, x_k^{(\ell)}; t_{1,k}, \dots, t_{\ell,k}), \\ \lim_{m \rightarrow \infty} (\xi_{k,m}^{(0)}, t_{j_1,k,m}\xi_{k,m}^{(1)}, \dots, t_{j_\ell,k,m}\xi_{k,m}^{(\ell)}) = (\xi_k^{(0)}, \xi_k^{(1)}, \dots, \xi_k^{(\ell)}). \end{aligned}$$

Then extracting a subsequence, we can find

$$\{(x_k^{(0)}, x_k^{(1)}, \dots, x_k^{(\ell)}; \xi_k^{(0)}, \xi_k^{(1)}, \dots, \xi_k^{(\ell)})\}_{k=1}^\infty \subset \text{SS}(F)$$

and $\{(t_{1,k}, \dots, t_{\ell,k})\}_{k=1}^\infty \subset (\mathbb{R}^+)^{\ell}$ such that

$$\begin{cases} \lim_{k \rightarrow \infty} (x_k^{(0)}, \frac{x_k^{(1)}}{t_{j_1,k}}, \dots, \frac{x_k^{(\ell)}}{t_{j_\ell,k}}) = (x^{(0)}, x^{(1)}, \dots, x^{(\ell)}), \\ \lim_{k \rightarrow \infty} (\xi_k^{(0)}, t_{j_1,k} \xi_k^{(1)}, \dots, t_{j_\ell,k} \xi_k^{(\ell)}) = (\xi^{(0)}, \xi^{(1)}, \dots, \xi^{(\ell)}), \\ \lim_{k \rightarrow \infty} (t_{1,k}, \dots, t_{\ell,k}) = (0, \dots, 0). \end{cases}$$

Since $\text{SS}(F)$ is conic, we have

$$(x_k^{(0)}, x_k^{(1)}, \dots, x_k^{(\ell)}; t_k \xi_k^{(0)}, t_k \xi_k^{(1)}, \dots, t_k \xi_k^{(\ell)}) \in \text{SS}(F),$$

where $t_k := \prod_{j=1}^{\ell} t_{j,k}$. Setting $c_{j,k} := \frac{1}{t_{j,k}}$ ($j = 1, \dots, \ell$) we obtain the desired result. \square

Remark 3.8. Theorem 3.6 extends the estimate of microsupport computed in [10].

4 Microfunctions along χ

In this section, we establish a vanishing theorem of cohomology groups for multi-microlocalization and introduce multi-microlocalized objects along χ which are natural extensions of sheaves of microfunctions and holomorphic ones.

4.1 The edge of the wedge theorem with bounds

We say that an open subset $\Omega \subset \mathbb{C}^n$ is an analytic open polyhedron if there exist holomorphic functions f_1, \dots, f_ℓ on \mathbb{C}^n satisfying

$$\Omega = \{z \in \mathbb{C}^n; |f_k(z)| < 1 \ (k = 1, 2, \dots, \ell)\}.$$

In the same way, a closed subset $K \subset \mathbb{C}^n$ is said to be an analytic closed polyhedron if K has the form

$$K = \{z \in \mathbb{C}^n; |f_k(z)| \leq 1 \ (k = 1, 2, \dots, \ell)\}.$$

for some holomorphic functions f_1, \dots, f_ℓ on \mathbb{C}^n . We first establish the edge of the wedge theorem for \mathcal{O}^w . Set $X := \mathbb{C}^n \times \mathbb{C}^m$.

Theorem 4.1. *Let Ω and ω in \mathbb{C}^n be relatively compact analytic open polyhedra, and let K be a closed analytic polyhedron or a closed convex subanalytic subset in \mathbb{C}^m . Then we have*

$$H^k(X; \mathbb{C}_{(\Omega \setminus \omega) \times K} \overset{w}{\otimes} \mathcal{O}_X) = 0 \quad (k \neq n).$$

To show the theorem, we need several lemmas. In what follows, we always assume that K is a closed analytic polyhedron or a closed convex subanalytic subset in \mathbb{C}^m . We note that, by the result of A. Dufresnoy [2], we have

$$H^k(\mathbb{C}^m; \mathbb{C}_K \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}^m}) = 0 \quad (k \neq 0)$$

for the both K .

Lemma 4.2. *Assume that Ω_i ($i = 1, 2, \dots, n$) is a non-empty open disk in \mathbb{C} . Set $\Omega := \Omega_1 \times \dots \times \Omega_n$. Then $R\Gamma(X; \mathbb{C}_{\Omega \times K} \overset{w}{\otimes} \mathcal{O}_X)$ is concentrated in degree n .*

Proof. We have $H^k(\mathbb{C}; \mathbb{C}_{\mathbb{C} \setminus \Omega_i} \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}}) = 0$ for $k \neq 0$, from which we get

$$H^k(\mathbb{C}; \mathbb{C}_{\Omega_i} \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}}) = 0 \quad (k \neq 1)$$

and the exact sequence

$$0 \rightarrow H^0(\mathbb{C}; \mathbb{C}_{\mathbb{C}} \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}}) \xrightarrow{\rho} H^0(\mathbb{C}; \mathbb{C}_{\mathbb{C} \setminus \Omega_i} \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}}) \rightarrow H^1(\mathbb{C}; \mathbb{C}_{\Omega_i} \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}}) \rightarrow 0$$

As ρ has a closed range by the maximum modulus principle for a holomorphic function, the cohomology group $H^1(\mathbb{C}; \mathbb{C}_{\Omega_i} \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}})$ becomes a FN space, and hence, we have an isomorphism in $D^b(FN)$

$$(4.1) \quad R\Gamma(\mathbb{C}; \mathbb{C}_{\Omega_i} \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}}) \simeq H^1(\mathbb{C}; \mathbb{C}_{\Omega_i} \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}})[-1].$$

Further, we have

$$H^k(\mathbb{C}^m; \mathbb{C}_K \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}^m}) = 0 \quad (k \neq 0),$$

and $H^0(\mathbb{C}^m; \mathbb{C}_K \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}^m})$ is a FN space to which $R\Gamma(\mathbb{C}^m; \mathbb{C}_K \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}^m})$ is isomorphic in $D^b(FN)$. Hence the claim of the lemma follows from the tensor product formula of Proposition 5.3 [8]. \square

The following lemma is a key in the proof of the theorem.

Lemma 4.3 (Martineau). *Let Ω_i and ω_i ($i = 1, 2, \dots, n$) be non-empty open disks in \mathbb{C} with $\omega_i \subset \Omega_i$. Set $\Omega = \Omega_1 \times \dots \times \Omega_n$ and $\omega = \omega_1 \times \dots \times \omega_n$. Then the canonical morphism associated with inclusion of sets*

$$\iota : H^n(X; \mathbb{C}_{\omega \times K} \overset{w}{\otimes} \mathcal{O}_X) \rightarrow H^n(X; \mathbb{C}_{\Omega \times K} \overset{w}{\otimes} \mathcal{O}_X)$$

is injective.

Remark 4.4. The above morphism does not have a closed range, and hence, the cohomology group $H^n(X; \mathbb{C}_{(\Omega \setminus \omega) \times K} \overset{w}{\otimes} \mathcal{O}_X)$ is not an FN space in general.

Proof. We apply the arguments in the proof of Theorem 4.1.6 [5] to our Whitney case. Let (z, w) be the coordinates of $X = \mathbb{C}_z^n \times \mathbb{C}_w^m$. Set, for $k = 1, \dots, n$ and for a subset $\alpha = \{i_1, \dots, i_\ell\}$ in the set $\{1, 2, \dots, n\}$,

$$\begin{aligned} V_\omega^{(k)} &:= \mathbb{C}_{z_1} \times \dots \times \mathbb{C}_{z_{k-1}} \times (\mathbb{C}_{z_k} \setminus \omega_k) \times \mathbb{C}_{z_{k+1}} \times \dots \times \mathbb{C}_n, \\ V_\omega^{(\alpha)} &:= V_\omega^{(i_1)} \cap \dots \cap V_\omega^{(i_\ell)} \text{ and } V_\omega^{(\emptyset)} := \mathbb{C}^n. \end{aligned}$$

We also define $V_\Omega^{(k)}$ and $V_\Omega^{(\alpha)}$ in the same way by replacing ω with Ω . Then \mathbb{C}_{ω_k} is isomorphic to the complex, in $D_{\mathbb{R}-c}^b(\mathbb{C})$,

$$\mathcal{L}_{\omega_k} : 0 \longrightarrow \mathbb{C}_{\mathbb{C}} \longrightarrow \mathbb{C}_{\mathbb{C} \setminus \omega_k} \longrightarrow 0.$$

Similarly we can define the complex \mathcal{L}_{Ω_k} which is isomorphic to \mathbb{C}_{Ω_k} in $D_{\mathbb{R}-c}^b(\mathbb{C})$. Then the canonical sheaf morphism $\mathbb{C}_{\mathbb{C} \setminus \omega_k} \rightarrow \mathbb{C}_{\mathbb{C} \setminus \Omega_k}$ induces the morphism of complexes $\mathcal{L}_{\omega_k} \rightarrow \mathcal{L}_{\Omega_k}$, which is nothing but an extension of the canonical sheaf morphism $\mathbb{C}_{\omega_k} \rightarrow \mathbb{C}_{\Omega_k}$ to the complexes. Now we have

$$\mathbb{C}_{\omega \times K} = \mathbb{C}_{\omega_1} \boxtimes_{\mathbb{C}} \dots \boxtimes_{\mathbb{C}} \mathbb{C}_{\omega_n} \boxtimes_{\mathbb{C}} \mathbb{C}_K \simeq \mathcal{L}_{\omega_1} \boxtimes_{\mathbb{C}} \dots \boxtimes_{\mathbb{C}} \mathcal{L}_{\omega_n} \boxtimes_{\mathbb{C}} \mathbb{C}_K,$$

and the last complex is isomorphic to the complex

$$\mathcal{L}_\omega : 0 \rightarrow \bigoplus_{|\alpha|=0} \mathbb{C}_{V_\omega^{(\alpha)} \times K} \rightarrow \bigoplus_{|\alpha|=1} \mathbb{C}_{V_\omega^{(\alpha)} \times K} \rightarrow \cdots \rightarrow \bigoplus_{|\alpha|=n} \mathbb{C}_{V_\omega^{(\alpha)} \times K} \rightarrow 0,$$

where $|\alpha|$ denotes the number of elements of a set α . By the same reasoning, $\mathbb{C}_{\Omega \times K}$ is isomorphic to the complex

$$\mathcal{L}_\Omega : 0 \rightarrow \bigoplus_{|\alpha|=0} \mathbb{C}_{V_\Omega^{(\alpha)} \times K} \rightarrow \bigoplus_{|\alpha|=1} \mathbb{C}_{V_\Omega^{(\alpha)} \times K} \rightarrow \cdots \rightarrow \bigoplus_{|\alpha|=n} \mathbb{C}_{V_\Omega^{(\alpha)} \times K} \rightarrow 0,$$

and the canonical sheaf morphism $\mathbb{C}_{V_\omega^{(\alpha)}} \rightarrow \mathbb{C}_{V_\Omega^{(\alpha)}}$ induces the one of complexes from \mathcal{L}_ω to \mathcal{L}_Ω . This morphism is an extension of the sheaf morphism $\mathbb{C}_{\omega \times K} \rightarrow \mathbb{C}_{\Omega \times K}$ to the complexes. It follows from the result in [2] that we get, for any $\alpha \subset \{1, 2, \dots, n\}$ and for $k \neq 0$,

$$H^k(\mathbb{C}^n; \mathbb{C}_{V_\omega^{(\alpha)}} \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}^n}) = H^k(\mathbb{C}^n; \mathbb{C}_{V_\Omega^{(\alpha)}} \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}^n}) = H^k(\mathbb{C}^m; \mathbb{C}_K \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}^m}) = 0,$$

and hence, we obtain

$$H^k(X; \mathbb{C}_{V_\omega^{(\alpha)} \times K} \overset{w}{\otimes} \mathcal{O}_X) = H^k(X; \mathbb{C}_{V_\Omega^{(\alpha)} \times K} \overset{w}{\otimes} \mathcal{O}_X) = 0 \quad (k \neq 0).$$

By these observations, we can conclude that the canonical morphism ι coincides with

$$\frac{H^0(X; \mathbb{C}_{V_\omega \times K} \overset{w}{\otimes} \mathcal{O}_X)}{\bigoplus_{|\alpha|=n-1} H^0(X; \mathbb{C}_{V_\omega^{(\alpha)} \times K} \overset{w}{\otimes} \mathcal{O}_X)} \rightarrow \frac{H^0(X; \mathbb{C}_{V_\Omega \times K} \overset{w}{\otimes} \mathcal{O}_X)}{\bigoplus_{|\alpha|=n-1} H^0(X; \mathbb{C}_{V_\Omega^{(\alpha)} \times K} \overset{w}{\otimes} \mathcal{O}_X)},$$

where the morphism of these cohomology groups is given by the natural restriction and we set $V_\omega := V_\omega^{\{1,2,\dots,n\}}$ and $V_\Omega := V_\Omega^{\{1,2,\dots,n\}}$ for simplicity.

Let us show injectivity of the above ι . We first note that $H^0(X; \mathbb{C}_{V_\omega \times K} \overset{w}{\otimes} \mathcal{O}_X)$ consists of holomorphic Whitney jets on $V_\omega \times K$, that is, a holomorphic Whitney jet is a family $F(z, w) = \{f_\beta(z, w)\}_{\beta \in \mathbb{Z}_{\geq 0}^{2m}}$ of continuous functions on $V_\omega \times K$ satisfying the conditions:

1. Every $f_\beta(z, w)$ is a holomorphic function of z in V_ω° for each $w \in K$. Furthermore $\partial_z^\tau f_\beta(z, w)$ ($\tau \in \mathbb{Z}_{\geq 0}^n$) is continuous in $V_\omega^\circ \times K$ and continuously extends to $V_\omega \times K$.

2. For any $\tau \in \mathbb{Z}_{\geq 0}^n$, the jet $F_\tau(z, w) := \{\partial_z^\tau f_\beta(z, w)\}_{\beta \in \mathbb{Z}_{\geq 0}^{2m}}$ satisfies Whitney's remainder term estimate with respect to variables $\operatorname{Re} w$ and $\operatorname{Im} w$ (i.e., with respect to indices $\beta \in \mathbb{Z}_{\geq 0}^{2m}$) which is locally uniform with respect to $z \in V_\omega$. It also satisfies $\bar{\partial}_{w_k} F = 0$ for $1 \leq k \leq m$.

Let $F(z, w) = \{f_\beta(z, w)\}_\beta \in H^0(X; \mathbb{C}_{V_\omega \times K} \overset{w}{\otimes} \mathcal{O}_X)$ be a holomorphic Whitney jet on $V_\omega \times K$. Let us define

$$G(z, w) := \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial\omega_1 \times \dots \times \partial\omega_n} \frac{F(\zeta, w)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} d\zeta,$$

where $\partial\omega_k$ is the boundary of ω_k with the clockwise orientation. Then, by repetition of the integration by parts, we can easily confirm that $G(z, w)$ satisfies the conditions 1. and 2. above, and hence, we obtain $G(z, w) \in H^0(X; \mathbb{C}_{V_\omega} \overset{w}{\otimes} \mathcal{O}_X)$. Let D be a sufficiently large disk in \mathbb{C} satisfying $z \in \operatorname{int} D^n$. It follows from Cauchy's integral formula that we get

$$F(z, w) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{(\partial\omega_1 - \partial D) \times \dots \times (\partial\omega_n - \partial D)} \frac{F(\zeta, w)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} d\zeta.$$

Hence $F(z, w) - G(z, w)$ is a sum of integrals of the form

$$\frac{1}{(2\pi\sqrt{-1})^n} \int_{\gamma_1 \times \dots \times \gamma_n} \frac{F(\zeta, w)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} d\zeta,$$

where γ_k is either $\partial\omega_k$ or $-\partial D$. Furthermore $\gamma_k = -\partial D$ holds at least one of k 's in the above $\gamma_1 \times \dots \times \gamma_n$. Hence these integrals are zero in $H^n(X; \mathbb{C}_{\omega \times K} \overset{w}{\otimes} \mathcal{O}_X)$. As a conclusion, the holomorphic Whitney jets $F(z, w)$ and $G(z, w)$ determine the same cohomology class in $H^n(X; \mathbb{C}_{\omega \times K} \overset{w}{\otimes} \mathcal{O}_X)$.

Now assume $\iota(F(z, w)) = 0$. Then, by deforming the path of the integration to $\partial\Omega_1 \times \dots \times \partial\Omega_n$, we have $G(z, w) = 0$ on $V_\Omega \times K$ since $F(z, w)$ belongs to $\bigoplus_{|\alpha|=n-1} H^0(X; \mathbb{C}_{V_\Omega^{(\alpha)} \times K} \overset{w}{\otimes} \mathcal{O}_X)$, i.e., a sum of holomorphic Whitney jets which are holomorphic on the entire \mathbb{C} with respect to some variable z_k . Hence $G(z, w) = 0$ on $V_\omega \times K$ follows from the unique continuation property of $G(z, w)$ with respect to the variables z . This shows the injectivity of ι . \square

As an immediate consequence of the above lemma, we obtain the following.

Lemma 4.5. *Let Ω and ω be the same as those given in the previous lemma. Then we have*

$$H^k(X; \mathbb{C}_{(\Omega \setminus \omega) \times K} \overset{w}{\otimes} \mathcal{O}_X) = 0 \quad (k < n).$$

Proof. From the previous two lemmas and the following long exact sequence, the result easily follows.

$$\begin{aligned} \rightarrow H^k(X; \mathbb{C}_{\omega \times K} \overset{w}{\otimes} \mathcal{O}_X) &\rightarrow H^k(X; \mathbb{C}_{\Omega \times K} \overset{w}{\otimes} \mathcal{O}_X) \\ &\rightarrow H^k(X; \mathbb{C}_{(\Omega \setminus \omega) \times K} \overset{w}{\otimes} \mathcal{O}_X) \rightarrow H^{k+1}(X; \mathbb{C}_{\omega \times K} \overset{w}{\otimes} \mathcal{O}_X) \rightarrow . \end{aligned}$$

□

Proof of Theorem 4.1. First we show the claim

$$H^k(X; \mathbb{C}_{(\Omega \setminus \omega) \times K} \overset{w}{\otimes} \mathcal{O}_X) = 0 \quad (k > n).$$

We have the isomorphism in $D^b(FN)$

$$R\Gamma(\mathbb{C}^m; \mathbb{C}_K \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}^m}) \simeq H^0(\mathbb{C}^m; \mathbb{C}_K \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}^m}).$$

Furthermore, in $D^b(FN)$, it follows from the definition that the object $R\Gamma(\mathbb{C}^n; \mathbb{C}_{\Omega \setminus \omega} \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}^n})$ is isomorphic to the complex of FN spaces of length n

$$\mathcal{L} : 0 \rightarrow \Gamma(\mathbb{C}^n; \mathbb{C}_{\Omega \setminus \omega} \overset{w}{\otimes} \mathcal{C}_{\mathbb{R}^{2n}}^{\infty, (0,0)}) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Gamma(\mathbb{C}^n; \mathbb{C}_{\Omega \setminus \omega} \overset{w}{\otimes} \mathcal{C}_{\mathbb{R}^{2n}}^{\infty, (0,n)}) \rightarrow 0.$$

Then, by Proposition 5.3 of [8], the object $R\Gamma(X; \mathbb{C}_{(\Omega \setminus \omega) \times K} \overset{w}{\otimes} \mathcal{O}_X)$ is isomorphic to the complex $\mathcal{L} \hat{\boxtimes}_{\mathbb{C}} H^0(\mathbb{C}^m; \mathbb{C}_K \overset{w}{\otimes} \mathcal{O}_{\mathbb{C}^m})$ of FN spaces of length n . Hence we have obtained the claim.

Now we show, for $k < n$, the k -th cohomology group vanishes. We may assume $\omega \subset \Omega$. Then there are holomorphic functions $f_1, \dots, f_{\ell'}, \dots, f_{\ell}$ on \mathbb{C}^n such that

$$\Omega = \{z \in \mathbb{C}^n; |f_1(z)| < 1, \dots, |f_{\ell'}(z)| < 1\}$$

and

$$\omega = \{z \in \mathbb{C}^n; |f_1(z)| < 1, \dots, |f_{\ell'}(z)| < 1, \dots, |f_{\ell}(z)| < 1\}.$$

As Ω is relative compact, there exists $R > 0$ such that

$$\Omega \subset \{z \in \mathbb{C}^n; |z_1| < R, \dots, |z_n| < R\}, \quad |f_k(\Omega)| < R \quad (k = 1, 2, \dots, \ell).$$

We now apply the well-known method due to Oka to the following situation. Let us consider the closed embedding $\varphi : X = \mathbb{C}_{z,w}^{n+m} \rightarrow \mathbb{C}_{z,\tau,w}^{n+\ell+m}$ defined by

$$\varphi(z, w) = (z, f_1(z), \dots, f_{\ell'}(z), \dots, f_{\ell}(z), w).$$

Set $\tilde{X} := \mathbb{C}_{z,\tau,w}^{n+\ell+m} = \mathbb{C}_{z,\tau}^{n+\ell} \times \mathbb{C}_w^m$. We also define $\tilde{\Omega}$ and $\tilde{\omega}$ in $\mathbb{C}_{z,\tau}^{n+\ell}$ by

$$\tilde{\Omega} := \left\{ (z, \tau) \in \mathbb{C}^{n+\ell}; \begin{array}{l} |z_1| < R, \dots, |z_n| < R, |\tau_1| < 1, \dots, |\tau_{\ell'}| < 1, \\ |\tau_{\ell'+1}| < R, \dots, |\tau_{\ell}| < R \end{array} \right\}$$

and

$$\tilde{\omega} := \left\{ (z, \tau) \in \mathbb{C}^{n+\ell}; \begin{array}{l} |z_1| < R, \dots, |z_n| < R, |\tau_1| < 1, \dots, |\tau_{\ell'}| < 1, \\ |\tau_{\ell'+1}| < 1, \dots, |\tau_{\ell}| < 1 \end{array} \right\}.$$

Noticing $(\Omega \setminus \omega) \times K = \varphi^{-1}((\tilde{\Omega} \setminus \tilde{\omega}) \times K)$, by Theorem 5.8 (ii) in [8], we have

$$\mathrm{R}\Gamma(X; \mathbb{C}_{(\Omega \setminus \omega) \times K} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) = \mathrm{R}\Gamma(X; \varphi^{-1}(\mathbb{C}_{(\tilde{\Omega} \setminus \tilde{\omega}) \times K} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\tilde{X}})).$$

Let \mathcal{M} be the right $\mathcal{D}_{\tilde{X}}$ module $\mathcal{D}_{\tilde{X}}/(\tau_1 - f_1(z), \dots, \tau_{\ell} - f_{\ell}(z))\mathcal{D}_{\tilde{X}}$. Then, as \mathcal{M} is defined globally on \tilde{X} and its support is contained in the graph of φ , we have

$$\mathrm{R}\Gamma(X; \varphi^{-1}(\mathbb{C}_{(\tilde{\Omega} \setminus \tilde{\omega}) \times K} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\tilde{X}})) \simeq \mathrm{R}\Gamma(\tilde{X}; \mathcal{M} \overset{L}{\otimes}_{\mathcal{D}_{\tilde{X}}} (\mathbb{C}_{(\tilde{\Omega} \setminus \tilde{\omega}) \times K} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\tilde{X}})).$$

Define the $\mathfrak{D} := \Gamma(\tilde{X}; \mathcal{D}_{\tilde{X}})$ -module \mathfrak{M} by $\mathfrak{D}/(\tau_1 - f_1(z), \dots, \tau_{\ell} - f_{\ell}(z))\mathfrak{D}$. Then we obtain

$$\mathrm{R}\Gamma(\tilde{X}; \mathcal{M} \overset{L}{\otimes}_{\mathcal{D}_{\tilde{X}}} (\mathbb{C}_{(\tilde{\Omega} \setminus \tilde{\omega}) \times K} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\tilde{X}})) \simeq \mathfrak{M} \overset{L}{\otimes}_{\mathfrak{D}} \mathrm{R}\Gamma(\tilde{X}; \mathbb{C}_{(\tilde{\Omega} \setminus \tilde{\omega}) \times K} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\tilde{X}}).$$

Since we have $\mathrm{H}^k(\tilde{X}; \mathbb{C}_{(\tilde{\Omega} \setminus \tilde{\omega}) \times K} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\tilde{X}}) = 0$ for $k < n + \ell$ by Lemma 4.5, and since \mathfrak{M} has a free \mathfrak{D} resolution of the length ℓ , we obtain, for $k < (n + \ell) - \ell = n$,

$$\mathrm{H}^k(\mathfrak{M} \overset{L}{\otimes}_{\mathfrak{D}} \mathrm{R}\Gamma(\tilde{X}; \mathbb{C}_{(\tilde{\Omega} \setminus \tilde{\omega}) \times K} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\tilde{X}})) = 0.$$

This completes the proof. \square

Corollary 4.6. *Let Ω , ω and K be the same as those in Theorem 4.1, and let $g : \mathbb{C}^n \rightarrow \mathbb{C}^d$ be a holomorphic map and L a closed analytic polyhedron in \mathbb{C}^d . Then we have, for $k < n - d$,*

$$H^k(X; \mathbb{C}_{((\Omega \setminus \omega) \cap g^{-1}(L)) \times K} \overset{w}{\otimes} \mathcal{O}_X) = 0.$$

Proof. The proof goes in the same way as that for Theorem 4.1. In this case, we apply Oka's method to the closed embedding $\varphi : \mathbb{C}^{n+m} \rightarrow \mathbb{C}^{n+\ell+d+m} =: \tilde{X}$ defined by

$$\varphi(z, w) := (z, f_1(z), \dots, f_\ell(z), g(z), w).$$

Then $((\Omega \setminus \omega) \cap g^{-1}(L)) \times K = \varphi^{-1}((\tilde{\Omega} \setminus \tilde{\omega}) \times L \times K)$ holds. It follows from Lemma 4.5 that we obtain, for $k < n + \ell$,

$$H^k(\tilde{X}; \mathbb{C}_{(\tilde{\Omega} \setminus \tilde{\omega}) \times L \times K} \overset{w}{\otimes} \mathcal{O}_{\tilde{X}}) = 0.$$

Hence, by noticing that the module \mathfrak{M} has a free resolution of length $\ell + d$, we have the conclusion. \square

The following theorem is well-known, which can be also proved by the same method as that in the proof of Theorem 4.1.

Theorem 4.7. *Let K_1 and K_2 be analytic compact polyhedra in \mathbb{C}^n and let U be a Stein open subanalytic subset in \mathbb{C}^m . Then we have*

$$H^k_{(K_1 \setminus K_2) \times U}(X; \mathcal{F}) = 0 \quad (k \neq n),$$

where \mathcal{F} is $\mathcal{O}_{X_{sa}}^t$ or \mathcal{O}_X .

4.2 Some geometrical preparations

Let X be \mathbb{R}^n with coordinates (x_1, \dots, x_n) and G a closed conic subset in X . Recall that G is said to be a proper cone with respect to direction $\xi \neq 0$ if $G \setminus \{0\} \subset \{x \in X; \langle x, \xi \rangle > 0\}$ holds. Note that the cone $G = \{0\}$ is proper

with respect to all the directions $\xi \neq 0$. We also say that G is a linear cone if there exist vectors ξ_1, \dots, ξ_k such that

$$G = \bigcap_{1 \leq i \leq k} \{x \in X; \langle x, \xi_i \rangle \geq 0\}.$$

If G is a proper cone, then G is linear if and only if there exist vectors $\xi_1, \dots, \xi_{k'}$ such that

$$G = \mathbb{R}_{\geq 0}\xi_1 + \dots + \mathbb{R}_{\geq 0}\xi_{k'} (= \overline{\mathbb{R}_{\geq 0}\xi_1 + \dots + \mathbb{R}_{\geq 0}\xi_{k'}}).$$

From the above equivalence the following lemma immediately follows.

Lemma 4.8. *Let G_j be closed subsets in X ($j = 1, \dots, \ell$). Assume that G_j 's are linear proper cones with respect to the same direction $\xi \neq 0$. Then $G_1 + \dots + G_\ell$ is a closed linear proper cone with respect to the direction ξ .*

Now let us back to the situation of the fiber formula in § 2.2. Let I_j be subsets in $\{1, \dots, n\}$ satisfying the conditions (1.3). Recall that $x^{(j)}$ denotes the coordinates x_i 's with $i \in \widehat{I_j}$ and $\xi^{(j)}$ designates the dual coordinates of $x^{(j)}$ ($j = 1, \dots, \ell$). We also denote by $x^{(0)}$ the coordinates (x_i) with $i \in \{1, \dots, n\} \setminus I$ where we set $I := \cup_j I_j$. Then the coordinates of S_χ^* is given by $(x^{(0)}; \xi^{(1)}, \dots, \xi^{(\ell)})$.

Let $p = (q; \xi) = (x^{(0)}; \xi^{(1)}, \dots, \xi^{(\ell)})$ be a point in S_χ^* . Recall also the definitions of $J_{\prec j}$, $J_{\succ j}$ and γ_j given in the fiber formula which are often used in subsequent arguments. Now we define the subset $J^*(\xi)$ of $\{1, \dots, \ell\}$ by

$$(4.2) \quad \{j \in \{1, \dots, \ell\}; \xi^{(\alpha)} = 0 \text{ for any } \alpha \in J_{\preceq j}\}.$$

Here $J_{\preceq j}$ denotes $J_{\prec j} \cup \{j\}$. In what follows, we suppose $I = \{1, \dots, n\}$ for simplicity. Hence the coordinates of S_χ^* is given by $(\xi^{(1)}, \dots, \xi^{(\ell)})$. Let $\xi = (\xi^{(1)}, \dots, \xi^{(\ell)})$ be a point in S_χ^* . Set

$$(4.3) \quad L(\xi) := \bigcap_{j \in J^*(\xi)} \{x \in \mathbb{R}^n; x^{(j)} = 0\}.$$

Lemma 4.9. *Let G_j ($j = 1, \dots, \ell$) be a closed linear cone with $G_j \setminus \{0\} \subset \gamma_j$ which has an interior point in γ_j if γ_j is non-empty ($j = 1, \dots, \ell$). Set $G := G_1 + \dots + G_\ell \subset \mathbb{R}^n$. Then we have the followings.*

1. *G is a closed linear cone which is proper with respect to some direction ξ_G .*
2. *We have $G \subset L(\xi)$ and $G = \overline{\text{Int}_{L(\xi)}(G)}$. In particular,*

$$R\mathcal{H}om_{\mathbb{C}_X}(\mathbb{C}_G, \mathbb{C}_X) = \mathbb{C}_{\text{Int}_{L(\xi)}(G)}[-\text{codim}_{\mathbb{R}}(L(\xi))]$$

holds where $\text{Int}_{L(\xi)}(G)$ denotes the set of interior points of G in $L(\xi)$.

Proof. Let us show the claim 1. of the lemma. Recall the definition of the cone V_j which appeared in the proof of Theorem 2.11. If $\xi^{(j)} \neq 0$, then $V_j \subset \mathbb{R}^n$ has the form

$$\{x = (x^{(k)}) \in X; x^{(j)} \in T_j, \delta|\langle x^{(j)}, \xi^{(j)} \rangle| \geq \sum_{k \in J_{\succ j}} |x^{(k)}|\}$$

where $\delta > 0$ and $T_j \subset \mathbb{R}^{n_j}$ is a proper closed convex cone with

$$T_j \subset \{x^{(j)} \in \mathbb{R}^{n_j}; \langle x^{(j)}, \xi^{(j)} \rangle \geq 0\} \text{ and } \xi^{(j)} \in \text{Int}_{\mathbb{R}^{n_j}} T_j.$$

If $\xi^{(j)} = 0$, then we set $V_j := \mathbb{R}^n$. It follows from the definitions of G_j and γ_j that G_j is contained in the polar cone V_j° for an appropriate V_j .

Let $\#J_{\succ j}$ denote the number of elements in $J_{\succ j}$. For $\sigma > 0$, we determine the positive real number σ_j by $\sigma^{\#J_{\succ j}}$. Now we define the vector by

$$(4.4) \quad \xi_G := (\sigma_1 \xi^{(1)}, \sigma_2 \xi^{(2)}, \dots, \sigma_\ell \xi^{(\ell)}).$$

Then, as $\#J_{\succ \alpha} < \#J_{\succ j}$ holds for $\alpha \in J_{\succ j}$, it follows the definition of V_j that the vector ξ_G belongs to the interior of V_j for any j by taking σ sufficiently large. Hence, since G_j is contained V_j° , we see that $G_j \setminus \{0\}$ is contained in $\{x \in \mathbb{R}^n; \langle x, \xi_G \rangle > 0\}$. Then the first claim is a consequence of Lemma 4.8.

Next we show the claim 2. of the lemma. By the definition of γ_j , we can easily see that $G \subset L(\xi)$ and G has an interior point in $L(\xi)$. Then, as G is convex, the rest of the claim follows from these. \square

4.3 The result for some family of real analytic submanifolds

Let $X = \mathbb{C}^n$ with coordinates $(z_1 = x_1 + \sqrt{-1}y_1, \dots, z_n = x_n + \sqrt{-1}y_n)$ and $\zeta_i = \xi_i + \sqrt{-1}\eta_i$ ($i = 1, \dots, n$) the dual variable of $z_i = x_i + \sqrt{-1}y_i$. Let I_j ($j = 1, 2, \dots, \ell$) be a subset of $\{1, 2, \dots, n\}$ which satisfies the conditions (1.3), and set I_0 by (1.8). Let $I_{\mathbb{R}}$ be a subset of $I := \bigcup_{1 \leq j \leq \ell} I_j$ and $I_{\mathbb{C}} := I \setminus I_{\mathbb{R}}$. Define, for $i \in I$, the function $q_i(z)$ in X by

$$q_i(z) := \begin{cases} z_i & (i \in I_{\mathbb{C}}), \\ \sqrt{-1} \operatorname{Im} z_i & (i \in I_{\mathbb{R}}). \end{cases}$$

Then we define the closed real analytic submanifolds

$$N_j := \{z \in X; q_i(z) = 0, i \in I_j\} \quad (j = 1, \dots, \ell),$$

and set

$$\chi := \{N_1, \dots, N_{\ell}\}, \quad N = N_1 \cap \dots \cap N_{\ell}.$$

In what follows, we regard the function q_i as the complex coordinate variable z_i if $i \in I_{\mathbb{C}}$ and as the imaginary coordinate variable $\sqrt{-1}y_i$ if $i \in I_{\mathbb{R}}$. In the same way, p_i is regarded as the dual variable of q_i , that is, p_i denotes ζ_i if $i \in I_{\mathbb{C}}$ and $\sqrt{-1}\eta_i$ if $i \in I_{\mathbb{R}}$. As usual convention, we write by $q^{(j)}$ (resp. $p^{(j)}$) the coordinates q_i 's (resp. p_i 's) with $i \in \widehat{I}_j$. Under these conventions, the coordinates of S_{χ}^* are given by

$$(q^{(0)}; p^{(1)}, \dots, p^{(\ell)}),$$

where $q^{(0)}$ denotes the set of the coordinate variables z_i 's ($i \in I_0$) and x_i 's ($i \in I_{\mathbb{R}}$). Let $\theta_* = (q_*; p_*) = (q_*^{(0)}; p_*^{(1)}, \dots, p_*^{(\ell)}) \in S_{\chi}^*$. Recall the definition of $J^*(\theta_*)$, that is,

$$\begin{aligned} J^*(\theta_*) &:= \{j \in \{1, \dots, \ell\}; p_*^{(\alpha)} = 0 \text{ for all } \alpha \in J_{\preceq j}\} \\ &= \{j \in \{1, \dots, \ell\}; p_*^{(\alpha)} = 0 \text{ for all } \alpha \text{ with } N_j \subset N_{\alpha}\}. \end{aligned}$$

We set

$$(4.5) \quad I^*(\theta_*) := \bigcup_{j \in J^*(\theta_*)} \widehat{I}_j \subset \{1, \dots, n\}.$$

Then we define the integer $N(\theta_*)$ by

$$(4.6) \quad N(\theta_*) = \#I + \#(I^*(\theta_*) \cap I_{\mathbb{C}}),$$

where $\#$ denotes the number of elements in a set. Note that $\#I$ is equal to $\text{Codim}_{\mathbb{C}} N$, i.e., the complex codimension of the maximal complex linear subspace contained in N .

Theorem 4.10. *We have*

$$H^k(\mu_{\chi}(\mathcal{O}_{X_{sa}}^w))_{\theta_*} = 0 \quad (k \neq N(\theta_*)).$$

Proof. We may assume $q_* = 0$. Furthermore, by a complex rotation of each variable z_i ($i \in I_{\mathbb{C}}$), we also assume that $p_*^{(j)}$ is purely imaginary or zero for $j = 1, \dots, \ell$. Set

$$J := \{1, \dots, \ell\}, \quad J' := J \setminus J^*(\theta_*), \quad J'' := J^*(\theta_*).$$

Recall $n_j = \#\widehat{I}_j$ and set

$$m_1 := \sum_{j \in J'} n_j, \quad m_2 := \sum_{j \in J''} n_j, \quad m_3 = n - m_1 - m_2.$$

Note that, because of $\sqcup_{j \in J} \widehat{I}_j = I$, we get $m_1 + m_2 = \#I$. Then we have $X = \mathbb{C}_{z'}^{m_1} \times \mathbb{C}_{z''}^{m_2} \times \mathbb{C}_{z'''}^{m_3}$ where the coordinates (z', z'', z''') are given by

$$(z', z'', z''') = (\{z^{(j)}\}_{j \in J'}, \{z^{(j)}\}_{j \in J''}, z^{(0)}).$$

Here $z^{(j)}$ (resp. $z^{(0)}$) denotes, as usual convention, the coordinates z_i 's with $i \in \widehat{I}_j$ (resp. $i \in I_0$). We also set

$$(q', q'') = (\{q^{(j)}\}_{j \in J'}, \{q^{(j)}\}_{j \in J''})$$

and

$$\begin{aligned} (p', p'') &= (\{p^{(j)}\}_{j \in J'}, \{p^{(j)}\}_{j \in J''}), \\ (p'_*, p''_*) &= (\{p_*^{(j)}\}_{j \in J'}, \{p_*^{(j)}\}_{j \in J''}). \end{aligned}$$

Note that, by the definition of $J^*(\theta_*)$, it follows that $p''_* = 0$ and $p'_* \neq 0$. Furthermore p'_* is purely imaginary by the assumption. Then define the partially complex linear subspaces $L \subset \mathbb{C}_{z'}^{m_1}$ and $Z \subset \mathbb{C}_{z''}^{m_2}$ by

$$\begin{aligned} L &:= \text{The linear subspace of } \mathbb{C}_{z'}^{m_1} \text{ spanned by the vectors } dq', \\ Z &:= \{z'' \in \mathbb{C}_{z''}^{m_2}; q'' = 0\}. \end{aligned}$$

Clearly the coordinates of L are given by q' . We denote by L^\perp the orthogonal linear subspace of L in $\mathbb{R}^{2m_1} = \mathbb{C}^{m_1}$. Then we have $\mathbb{C}^{m_1} = L^\perp \times L$ and

$$L^\perp \subset \mathbb{R}^{m_1} = \mathbb{R}^{m_1} \times \{0\} \subset \mathbb{R}^{m_1} \times \sqrt{-1}\mathbb{R}^{m_1} = \mathbb{C}^{m_1}$$

as q_i is either z_i or $\sqrt{-1}y_i$. Now let us define the closed set γ_j ($j \in J'$) in L which corresponds to the one in the fiber formula.

$$(4.7) \quad \gamma_j := \left\{ q' = (q^{(k)})_{k \in J'} \in L; \begin{array}{ll} q^{(k)} = 0 & (k \in J' \text{ with } k \in J_{\prec j} \sqcup J_{\# j}) \\ \operatorname{Re}\langle q^{(k)}, p_*^{(k)} \rangle > 0 & (k = j) \end{array} \right\}.$$

It follows from the fiber formula that we have

$$H^k(\mu_\chi(\mathcal{O}_{X_{sa}}^w))_{\theta_*} = \varinjlim_{G, U_1, U_2, U_3} H^k_{(L^\perp \times G) \times Z \times \mathbb{C}^{m_3}} (U_1 \times U_2 \times U_3; \mathcal{O}_{X_{sa}}^w).$$

Here U_i runs through a family of open convex subanalytic neighborhoods of the origin in \mathbb{C}^{m_i} ($i = 1, 2, 3$) and $G \subset L$ runs through a family of closed subanalytic cones of the form $G := \sum_{j \in J'} G_j$ with a subanalytic closed cone $G_j \subset L$ satisfying $G_j \setminus \{0\} \subset \gamma_j$. Here we may assume that each G_j is a proper linear cone in L and it has an interior point in γ_j . Hence, by Lemma 4.9, the G is a closed proper linear cone in L with some direction. Furthermore, the $L^\perp \times G$ and $L^\perp \times \operatorname{Int}_L(G)$ are analytic polyhedra in $\mathbb{C}_{z'}^{m_1}$ and the latter one is a non-empty open cone. By the claim 2. of the same lemma, we have

$$H^k_{(L^\perp \times G) \times Z \times \mathbb{C}^{m_3}} (U_1 \times U_2 \times U_3; \mathcal{O}_{X_{sa}}^w) = H^{k-m'_2} (X; \mathbb{C}_{W(G, U_1, U_2, U_3)} \overset{w}{\otimes} \mathcal{O}_X),$$

where

$$(4.8) \quad m'_2 := m_2 + \#(I^*(\theta_*) \cap I_{\mathbb{C}})$$

and

$$W(G, U_1, U_2, U_3) := ((L^\perp \times \operatorname{Int}_L(G)) \cap \overline{U_1}) \times (Z \cap \overline{U_2}) \times \overline{U_3}.$$

From now on, we also use as coordinates variables of $\mathbb{C}_{z'}^{m_1}$

$$z' = (z_{i_1}, \dots, z_{i_{m_1}}) = (x_{i_1} + \sqrt{-1}y_{i_1}, \dots, x_{i_{m_1}} + \sqrt{-1}y_{i_{m_1}}).$$

Furthermore, we identify \mathbb{R}^{m_1} and $\sqrt{-1}\mathbb{R}^{m_1}$ with the linear subspaces $\mathbb{R}^{m_1} \times \sqrt{-1}\{0\}$ and $\{0\} \times \sqrt{-1}\mathbb{R}^{m_1}$ in $\mathbb{C}^{m_1} = \mathbb{R}^{m_1} \times \sqrt{-1}\mathbb{R}^{m_1}$ respectively. Since $p'_* \neq 0$ and it is purely imaginary as we have already noted, the cone

$$\tilde{G} := \frac{1}{\sqrt{-1}}(G \cap \sqrt{-1}\mathbb{R}^{m_1})$$

is proper and linear in \mathbb{R}^{m_1} . Furthermore, as p'_* is purely imaginary, we may assume

$$G \subset (L \cap \mathbb{R}^{m_1}) \times \sqrt{-1}\tilde{G} \subset L.$$

By Lemma 4.9, we choose a vector $\xi_{\tilde{G}} \in \mathbb{R}^{m_1}$ with $|\xi_{\tilde{G}}| = 1$ satisfying

$$\tilde{G} \subset \{x' \in \mathbb{R}^{m_1}; \langle x', \xi_{\tilde{G}} \rangle > 0\} \cup \{0\}.$$

Let us define the holomorphic function on $\mathbb{C}_{z'}^{m_1}$ by

$$\varphi(z') := \langle z', \xi_{\tilde{G}} \rangle + \sqrt{-1} \sum_{1 \leq k \leq m_1} z_{i_k}^2.$$

Let $a > 0$ and ϵ be sufficiently small positive real numbers, and we set

$$K := (Z \cap \overline{U_2}) \times \overline{U_3} \subset \mathbb{C}^{m_2} \times \mathbb{C}^{m_3},$$

$$\Omega := (L^\perp \times \text{Int}_L(G)) \cap \left\{ z' \in \mathbb{C}^{m_1}; |z_{i_k}| < a (1 \leq k \leq m_1) \right\},$$

$$\omega := (L^\perp \times \text{Int}_L(G)) \cap \left\{ z' \in \mathbb{C}^{m_1}; \text{Im } \varphi(z') > \epsilon, |z_{i_k}| < a (1 \leq k \leq m_1) \right\}.$$

We take an open poly-disks with center at the origin as U_i ($i = 1, 2, 3$). Then K is a convex closed analytic polyhedron in $\mathbb{C}^{m_2} \times \mathbb{C}^{m_3}$, and Ω and ω are relatively compact and analytic open polyhedra in \mathbb{C}^{m_1} . Hence it follows from Theorem 4.1 that we have

$$(4.9) \quad H^k(X; \mathbb{C}_{(\Omega \setminus \omega) \times K} \overset{\text{w}}{\otimes} \mathcal{O}_X) = 0 \quad (k \neq m_1).$$

Now, by applying the same reasoning as that in the proof of Theorem 2.2.2 [6], we obtain, for each G and a sufficiently small $a > 0$,

$$(4.10) \quad \begin{aligned} & \varinjlim_{U_1, U_2, U_3} H^{k-m'_2} \left(X; \mathbb{C}_{W(G, U_1, U_2, U_3)} \overset{\text{w}}{\otimes} \mathcal{O}_X \right) \\ &= \varinjlim_{\epsilon > 0, U_2, U_3} H^{k-m'_2} (X; \mathbb{C}_{(\Omega \setminus \omega) \times K} \overset{\text{w}}{\otimes} \mathcal{O}_X). \end{aligned}$$

As a matter of fact, as \tilde{G} is proper with respect to the direction $\xi_{\tilde{G}}$, we get

$$\tilde{G} \subset \{y' \in \mathbb{R}^{m_1}; |y' - \langle y', \xi_{\tilde{G}} \rangle \xi_{\tilde{G}}| \leq \sigma \langle y', \xi_{\tilde{G}} \rangle\}$$

for some $\sigma > 0$. Hence there exists $a_\sigma > 0$ depending only on σ such that

$$\sum_{1 \leq k \leq m_1} y_{i_k}^2 \leq \frac{1}{2} \langle y', \xi_{\tilde{G}} \rangle \quad (|y_{i_1}|, \dots, |y_{i_m}| \leq a_\sigma, y' \in \tilde{G}).$$

Then, by noticing

$$\text{Im } \varphi(z') = \langle y', \xi_{\tilde{G}} \rangle + \sum_{1 \leq k \leq m_1} (x_{i_k}^2 - y_{i_k}^2),$$

we obtain

$$\frac{1}{2} \langle y', \xi_{\tilde{G}} \rangle + \sum_{1 \leq k \leq m_1} x_{i_k}^2 \leq \epsilon \quad (0 < a \leq a_\sigma, z' \in \Omega \setminus \omega).$$

Therefore, if we take $\epsilon > 0$ sufficiently small, by noticing that \tilde{G} is proper with respect to the direction $\xi_{\tilde{G}}$, the subset $W(G, U_1, U_2, U_3)$ contains $(\Omega \setminus \omega) \times K$ as a closed subset. On the other hand, clearly the subset $\{z' \in \mathbb{C}^{m_1}; \text{Im } \varphi(z') \leq \epsilon\}$ ($\epsilon > 0$) is a neighborhood of $z' = 0$, the subset $(\Omega \setminus \omega) \times K$ contains $W(G, U', U_2, U_3)$ for a sufficiently small open neighborhood U' of the origin in $\mathbb{C}_{z'}^{m_1}$ as a closed subset. Hence we have a conclusion (4.10).

Then the result follows from (4.9) and

$$m_1 + m'_2 = m_1 + m_2 + \#(I^*(\theta_*) \cap I_{\mathbb{C}}) = \#I + \#(I^*(\theta_*) \cap I_{\mathbb{C}}) = N(\theta_*).$$

This completes the proof. \square

We also have the similar results for \mathcal{O}_X^t and \mathcal{O}_X by employing the same argument as that in the above proof, which is much easier because we can take $G \times Z$ as a cone G in the proof,

Theorem 4.11. *We have*

$$H^k(\mu_\chi(\mathcal{F}))_{\theta_*} = 0 \quad (k \neq \text{codim}_{\mathbb{C}} N = \#I),$$

where \mathcal{F} is either $\mathcal{O}_{X_{sa}}^t$ or \mathcal{O}_X .

4.4 The typical examples

As the results given in the previous subsection has been considered in a fairly general situation, we here describe the corresponding results for typical cases.

We first consider the corresponding result for families of complex submanifolds, i.e., $I = I_{\mathbb{C}}$. Let X be a complex manifold and $\chi = \{Z_1, \dots, Z_\ell\}$ a family of closed complex submanifolds of X which satisfies the conditions H1, H2 and H3. Set $Z = Z_1 \cap \dots \cap Z_\ell$. Let $p = (q; \zeta) = (q; \zeta^{(1)}, \dots, \zeta^{(\ell)}) \in S_\chi^*$. Remember that the subset $J^*(p)$ of $\{1, \dots, \ell\}$ was defined by

$$J^*(p) := \{j \in \{1, \dots, \ell\}; \zeta^{(\alpha)} = 0 \text{ for all } \alpha \text{ with } Z_j \subset Z_\alpha\}.$$

We also define $\hat{J}^*(p)$ by the subset of $J^*(p)$ that consists of the minimal elements with respect to the order relation $k \prec j \iff Z_k \subsetneq Z_j$ for $k, j \in J^*(p)$. Now we define the integer $N(p)$ by

$$(4.11) \quad N(p) = \text{codim}_{\mathbb{C}} Z + \sum_{j \in \hat{J}^*(p)} \text{codim}_{\mathbb{C}} Z_j.$$

Then the following theorem immediately comes from Theorem 4.10.

Corollary 4.12. *We have*

$$H^k(\mu_\chi(\mathcal{O}_{X_{sa}}^w))_p = 0 \quad (k \neq N(p)).$$

Remark 4.13. In the complex case, the result depends on a point $p \in S_\chi^*$. For example, $N(p) = 2 \text{codim}_{\mathbb{C}} Z$ if $p = (q; 0, \dots, 0)$ and $N(p) = \text{codim}_{\mathbb{C}} Z$ if all the $\zeta^{(j)}$'s are non-zero.

Corollary 4.14. *We have*

$$H^k(\mu_\chi(\mathcal{F}))_p = 0 \quad (k \neq \text{codim}_{\mathbb{C}} Z),$$

where \mathcal{F} is \mathcal{O}_X^t or \mathcal{O}_X .

Definition 4.15. The sheaf of holomorphic microfunctions along χ in S_χ^* is defined by

$$(4.12) \quad \mathcal{C}_\chi^\mathbb{R} := \mu_\chi(\mathcal{O}_X) \otimes_{\mathbb{Z}_{S_\chi^*}} \text{or}_{S_\chi^*}[\text{codim}_\mathbb{C} Z],$$

where $\text{or}_{S_\chi^*}$ denotes the orientation sheaf of S_χ^* . We also define $\mathcal{C}_\chi^{\mathbb{R},f}$ and $\mathcal{C}_\chi^{\mathbb{R},w}$ by replacing \mathcal{O}_X in the above definition with \mathcal{O}_X^t and \mathcal{O}_X^w respectively.

Note that $\mathcal{C}_\chi^\mathbb{R}$ and $\mathcal{C}_\chi^{\mathbb{R},f}$ are really sheaves on S_χ^* . We also note that $\mathcal{C}_\chi^{\mathbb{R},w}$ is a complex. It is, however, concentrated in degree 0 outside the zero section, i.e., $\{(z; \zeta^{(1)}, \dots, \zeta^{(\ell)}) \in S_\chi^*; \zeta^{(j)} \neq 0\}$.

Next we consider the corresponding result for the case $I = I_\mathbb{R}$. Let M be a connected real analytic manifold and X its complexification. Let $\Theta_1, \dots, \Theta_\ell$ be real analytic vector subbundles of TM which are involutive, that is, $[\theta_1, \theta_2] \in \Theta_k$ for any vector fields $\theta_1, \theta_2 \in \Theta_k$. We denote by $\Theta_k^\mathbb{C} \subset TX$ the complex vector subbundle over X that is a complexification of Θ_k near M . Now we introduce the conditions for Θ_k 's which are counterparts of the ones H1, H2 and H3. Set, for $1 \leq k \leq \ell$,

$$\text{NR}(k) := \{j \in \{1, \dots, \ell\}; \Theta_j \not\subseteq \Theta_k, \Theta_k \not\subseteq \Theta_j\}$$

Then we assume that, for any $q \in M$ and any k with $\text{NR}(k) \neq \emptyset$,

$$(TM)_q = (\Theta_k)_q + \left(\bigcap_{j \in \text{NR}(k)} (\Theta_j)_q \right).$$

We also assume that, for simplicity, Θ_k 's are mutually distinct, i.e., $\Theta_{k_1} \neq \Theta_{k_2}$ if $k_1 \neq k_2$. For a local description of Θ_k 's, we have the lemma below.

Lemma 4.16. *Under the above situation, there exist subsets I_j ($j = 1, \dots, \ell$) of $\{1, \dots, n\}$ satisfying the condition (1.3) for which the following holds. For every $q \in M$, there exist an open neighborhood $U \subset M$ of q and a real analytic coordinates system (x_1, \dots, x_n) of U such that, in U , each Θ_k is given by*

$$\{(x; \nu) \in T\mathbb{R}^n; \nu_i = 0 (i \in I_k)\}.$$

Proof. As Θ_k is involutive, there exist locally real analytic functions $\pi_i^{(k)}$ ($i = 1, \dots, \text{codim}_{\mathbb{R}} \Theta_k$) such that

$$\Theta_k = \{(q; \nu) \in TM; d\pi_i^{(k)}(q)(\nu) = 0 (i = 1, \dots, \text{codim}_{\mathbb{R}} \Theta_k)\}.$$

Then, employing the same argument as that in Proposition 1.2 in [4], we can choose desired coordinate functions from these $\pi_i^{(k)}$'s. \square

Let $N_{M,j} \subset X$ ($j = 1, \dots, \ell$) be the union of the complex integral submanifolds of the involutive complex vector bundle $\Theta_j^{\mathbb{C}} \subset TX$ passing through each point $q \in M$, that is,

$$N_{M,j} := \bigcup_{q \in M} \mathcal{L}(\Theta_j^{\mathbb{C}}, q)$$

where $\mathcal{L}(\Theta_j^{\mathbb{C}}, q)$ denotes the complex integral submanifold of Θ_j passing through the point q . Set

$$\chi := \{N_{M,1}, \dots, N_{M,\ell}\}, \quad N_M := N_{M,1} \cap \dots \cap N_{M,\ell} \subset X.$$

These $N_{M,j}$'s and S_{χ}^* are locally described as follows: Let

$$(z_1 = x_1 + \sqrt{-1}y_1, \dots, z_n = x_n + \sqrt{-1}y_n).$$

be a system of local coordinates of X which are complexification of the coordinates (x_1, \dots, x_n) given in Lemma 4.4. Then it follows from the lemma that we have

$$N_{M,j} = \{z \in X; \text{Im } z_i = 0 (i \in I_j)\} \quad (j = 1, \dots, \ell),$$

where I_j 's were given in the lemma. We also set $I_0 = \widehat{I}_0$ by (1.8). Denote by $z^{(j)} = x^{(j)} + \sqrt{-1}y^{(j)}$ the coordinates variables $z_i = x_i + \sqrt{-1}y_i$'s with $i \in \widehat{I}_j$ and by $\zeta^{(j)} = \xi^{(j)} + \sqrt{-1}\eta^{(j)}$ the dual variables of $z^{(j)} = x^{(j)} + \sqrt{-1}y^{(j)}$ for $j = 0, \dots, \ell$. Then the coordinates of S_{χ}^* are locally given by

$$(z^{(0)}, x^{(1)}, \dots, x^{(\ell)}; \sqrt{-1}\eta^{(1)}, \dots, \sqrt{-1}\eta^{(\ell)}).$$

Corollary 4.17. *Let $p \in S_{\chi}^*$. Then we have*

$$H^k(\mu_{\chi}(\mathcal{F}))_p = 0 \quad (k \neq \text{codim}_{\mathbb{R}} N_M),$$

where \mathcal{F} is either $\mathcal{O}_{X_{sa}}^w$, $\mathcal{O}_{X_{sa}}^t$ or \mathcal{O}_X .

Remark 4.18. In this case, the result is independent of a point $p \in S_\chi^*$ contrary to the complex case.

Definition 4.19. The sheaf of microfunctions along χ with holomorphic parameters is defined by

$$(4.13) \quad \mathcal{C}_{N_M, \chi} := \mu_\chi(\mathcal{O}_X) \otimes_{\mathbb{Z}_{S_\chi^*}} \text{or}_{S_\chi^*}[\text{codim}_{\mathbb{R}} N_M],$$

where $\text{or}_{S_\chi^*}$ denotes the orientation sheaf of S_χ^* . We also define $\mathcal{C}_{N_M, \chi}^f$ and $\mathcal{C}_{N_M, \chi}^w$ by replacing \mathcal{O}_X in the above definition with \mathcal{O}_X^t and \mathcal{O}_X^w respectively.

Note that these are really sheaves in S_χ^* , that is, they are concentrated in degree 0 everywhere.

5 Applications to \mathcal{D} -modules

In this section, we consider applications of multi-microlocalizations to \mathcal{D} -module theory. First, recall the notation of § 2.1; for example, let $\tau_i: E_i \rightarrow Z$ ($1 \leq i \leq \ell$) be vector bundles over Z , and let E_i^* be the dual bundle of E_i .

Theorem 5.1 (cf. [16]). *Let F be a multi-conic object on E . Then there exists a natural isomorphism*

$$\tau^! R\tau_1 F \simeq Rp_{1*} p_2^!(F^{\wedge_E}),$$

and the natural morphism $F \rightarrow \tau^! R\tau_1 F$ is embedded to the following distinguished triangle:

$$F \rightarrow \tau^! R\tau_1 F \rightarrow Rp_{1*}^+ p_2^{+!}(F^{\wedge_E}) \xrightarrow{+1}.$$

Proof. If $\ell = 1$, the results follows by Lemma A.2 of [16]. Assume $\ell > 1$, and set $E' := \times_{Z, i=2}^{\ell} E_i$, $E'^* := \times_{Z, i=2}^{\ell} E_i^*$, and $P'_{E'} := \times_{Z, i=2}^{\ell} P'_i$. Moreover, let $\wedge_{E'}$

(resp $\vee_{E'}^*$) be the composition of \wedge_i (resp. \vee_i^*) for $i = 2, \dots, \ell$. Consider:

$$\begin{array}{ccccc}
 & & & p_2 & \\
 & & & \curvearrowright & \\
 & & E \times_Z E^* & \xrightarrow{p_{E',2}} & E_1 \times_Z E^* \xrightarrow{p_{1,2}} E^* \\
 & \nearrow p_1 & \downarrow p_{1,1} & \square & \downarrow p_{1,1} \\
 E & \xleftarrow{p_{E',1}} & E_1 \times_Z E' \times_Z E'^* & \xrightarrow{p_{E',2}} & E_1 \times_Z E'^*
 \end{array}$$

By the commutative diagram

$$\begin{array}{ccc}
 E & \xrightarrow{\tau_1} & E' \\
 \downarrow \tau_{E'} & \square & \downarrow \tau_{E'} \\
 E_1 & \xrightarrow{\tau_1} & Z
 \end{array}$$

we have

$$\begin{aligned}
 \tau_{E'}^! R\tau_{E'!} \tau_1^! R\tau_{1!} F &= \tau_{E'}^! R\tau_{E'!} \tau_1^{-1} R\tau_{1!} F \otimes \omega_{E/Z} \\
 &= \tau_{E'}^! \tau_1^{-1} R\tau_{E'!} R\tau_{1!} F \otimes \omega_{E/Z} = \tau_{E'}^! \tau_1^! R\tau_{E'!} R\tau_{1!} F = \tau^! R\tau_! F.
 \end{aligned}$$

Hence, by Lemma A.2 of [16] and induction hypothesis, we obtain:

$$\begin{array}{ccc}
 F & \xrightarrow{\sim} & F^{\wedge_{E'} \vee_{E'}^*} \xrightarrow{\sim} F^{\wedge_{E'} \vee_{E'}^* \wedge_1 \vee_1^*} \\
 \downarrow & & \downarrow \\
 \tau_{E'}^! R\tau_{E'!} F & \xrightarrow{\sim} & Rp_{E',1*} p_{E',2}^! (F^{\wedge_{E'}}) \\
 \downarrow & & \downarrow \\
 \tau_1^! R\tau_{1!} \tau_{E'}^! R\tau_{E'!} F & \xrightarrow{\sim} & \tau_1^! R\tau_{1!} Rp_{E',1*} p_{E',2}^! (F^{\wedge_{E'}}) \\
 \parallel & & \downarrow \wr \\
 \tau^! R\tau_! F & \xrightarrow{\sim} & Rp_{1,1*} p_{1,2}^! ((Rp_{E',1*} p_{E',2}^! (F^{\wedge_{E'}}))^{\wedge_1})
 \end{array} \tag{5.1}$$

For the same reasoning as in (2.1), for any multi-conic object H on $E_1 \times_Z E'^*$ we have

$$H^{\vee_i^* \vee_1} = H^{\vee_1 \vee_i^*}.$$

Therefore, we have

$$(5.2) \quad H^{\vee_i^* \wedge_1} = (H^{\vee_i^* \vee_1})^{\text{id} \times a} \otimes \omega_{E_1/Z}^{\otimes -1} = (H^{\vee_1 \vee_i^*})^{\text{id} \times a} \otimes \omega_{E_1/Z}^{\otimes -1} = H^{\wedge_1 \vee_i^*}$$

Thus, by Proposition 2.2 (2) and (5.2) we have

$$(5.3) \quad F^{\wedge_{E'} \vee_{E'}^* \wedge_1 \vee_1^*} = Rp'_{1*} p_2'^{\dagger}(F^{\wedge_E}).$$

Lastly, by Proposition 3.7.13 of [7] we have

$$(5.4) \quad \begin{aligned} & Rp_{1,1*} p_{1,2}^{\dagger}((Rp_{E',1*} p_{E',2}^{\dagger}(F^{\wedge_{E'}}))^{\wedge_1}) \\ &= Rp_{1,1*} p_{1,2}^{\dagger} Rp_{E',1*}((p_{E',2}^{\dagger}(F^{\wedge_{E'}}))^{\wedge_1}) \\ &= Rp_{1,1*} p_{1,2}^{\dagger} Rp_{E',1*} p_{E',2}^{\dagger}(F^{\wedge_{E'} \wedge_1}) = Rp_{1*} p_2^{\dagger}(F^{\wedge_E}). \end{aligned}$$

Therefore, by (5.1), (5.3), (5.4) and the definition of P^+ , we have

$$\begin{array}{ccccc} F & \xrightarrow{\quad} & \tau^{\dagger} R\tau_! F & & \\ \downarrow \wr & & \downarrow \wr & & \\ F^{\wedge_{E'} \vee_{E'}^* \wedge_1 \vee_1^*} & \xrightarrow{\quad} & Rp_{1,1*} p_{1,2}^{\dagger}((Rp_{E',1*} p_{E',2}^{\dagger}(F^{\wedge_{E'}}))^{\wedge_1}) & & \\ \downarrow \wr & & \downarrow \wr & & \\ Rp'_{1*} p_2'^{\dagger}(F^{\wedge_{E'}}) & \xrightarrow{\quad} & Rp_{1*} p_2^{\dagger}(F^{\wedge_{E'}}) & \xrightarrow{\quad} & Rp_{1*}^+ p_2^{\dagger}(F^{\wedge_E}) \xrightarrow{+1}. \end{array}$$

The commutativity follows from the constructions. \square

Therefore, we obtain the following:

Theorem 5.2. *Let X be a real analytic manifold, and assume that the family $\chi = \{M_i\}_{i=1}^{\ell}$ of submanifolds in X satisfies conditions H1, H2 and H3. Set $M := \bigcap_{i=1}^{\ell} M_i$. Then for any $F \in D^b(k_{X_{\text{sa}}})$, there exists the following distinguished triangle:*

$$(5.5) \quad \nu_{\chi}(F) \rightarrow \tau^{-1} R\Gamma_Y(F) \otimes \omega_{M/X}^{\otimes -1} \rightarrow Rp_{1*}^+(p_2^+)^{-1} \mu_{\chi}(F) \otimes \omega_{M/X}^{\otimes -1} \xrightarrow{+1}.$$

Proof. By the definition of multi-microlocalization and Theorem 5.1 we have (see also Remark 2.1)

$$\nu_{\chi}(F) \rightarrow \tau^{\dagger} R\tau_! \nu_{\chi}(F) \rightarrow Rp_{1*}^+ p_2^{\dagger} \mu_{\chi}(F) \xrightarrow{+1}.$$

Since τ and p_2^+ are projections, we have

$$\begin{aligned} \tau^{\dagger} &\simeq \tau^{-1} \otimes \omega_{S_{\chi}/M} \simeq \tau^{-1} \otimes \omega_{M/X}^{\otimes -1}, \\ p_2^{\dagger} &\simeq (p_2^+)^{-1} \otimes \omega_{P^+/M} \simeq (p_2^+)^{-1} \otimes \omega_{M/X}^{\otimes -1}. \end{aligned}$$

Hence we prove the theorem. \square

By Theorem 3.6, under the identifications $T^*S_\chi^* = T^*S_\chi = S_{\chi^*}$, we have

$$\text{SS}(\mu_\chi(F)) = \text{SS}(\nu_\chi(F)) \subset C_{\chi^*}(\text{SS}(F)).$$

In particular we obtain

$$(5.6) \quad \text{supp } \mu_\chi(F) \subset S_\chi^* \cap C_{\chi^*}(\text{SS}(F)).$$

Thus we obtain:

Corollary 5.3. *If $\dot{S}_\chi^* \cap C_{\chi^*}(\text{SS}(F)) = \emptyset$, then*

$$\nu_\chi(F) \simeq \tau^{-1} R\Gamma_Y(F) \otimes \omega_{M/X}^{\otimes -1}.$$

Now, let $\chi = \{Y_i\}_{i=1}^\ell$, and assume that each Y_i and $Y := \bigcap_{i=1}^\ell Y_i$ are complex submanifolds of X . As usual, let \mathcal{D}_X be the sheaf of holomorphic differential operators on X . Let \mathcal{M} be a coherent \mathcal{D}_X -module, and $\text{Ch } \mathcal{M}$ the characteristic variety of \mathcal{M} . Then, for $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$, it is known that $\text{SS}(F) = \text{Ch } \mathcal{M}$. From (5.5), we have

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{O}_X)) &\rightarrow \tau^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_Y(\mathcal{O}_X)) \otimes \omega_{Y/X}^{\otimes -1} \\ &\rightarrow Rp_{1*}^+(p_2^+)^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{O}_X)) \otimes \omega_{Y/X}^{\otimes -1} \xrightarrow{+1}. \end{aligned}$$

Let $f: Y \hookrightarrow X$ be the canonical embedding. We can define the following natural mappings associated with f :

$$T^*Y \xleftarrow{f_d} Y \times_X T^*X \xrightarrow{f_\pi} T^*X.$$

We define the *inverse image* of \mathcal{M} by

$$Df^*\mathcal{M} := \mathcal{O}_Y \overset{L}{\otimes}_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{M}.$$

Assume that Y is non-characteristic for \mathcal{M} ; that is, $T_Y^*X \cap \text{Ch } \mathcal{M} \subset T_X^*X$. Then, it is known that $Df^*\mathcal{M}$ is identified with $Df^*\mathcal{M} := H^0 Df^*\mathcal{M}$, and $Df^*\mathcal{M}$ is a coherent \mathcal{D}_Y -module.

Theorem 5.4. *Assume that Y is non-characteristic for \mathcal{M} . Then*

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{O}_X)) &\simeq \tau^{-1} R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{O}_Y) \\ &\simeq \tau^{-1} f^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X). \end{aligned}$$

Proof. By the non-characteristic condition and Cauchy-Kovalevskaja-Kashiwara theorem, we obtain the following isomorphisms:

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_Y(\mathcal{O}_X)) \otimes \omega_{Y/X}^{\otimes -1} &\simeq R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{O}_Y) \\ &\simeq f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X). \end{aligned}$$

Hence by Corollary 5.3, we may show $\dot{S}_\chi^* \cap C_{\chi^*}(\text{Ch } \mathcal{M}) = \emptyset$. Assume that there exists a point

$$(x^{(0)}; \xi^{(1)}, \dots, \xi^{(\ell)}) \in \dot{S}_\chi^* \cap C_{\chi^*}(\text{Ch } \mathcal{M}).$$

Then by Theorem 3.7 there exist sequences

$$\begin{aligned} \{(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}; \xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)})\}_{n=1}^\infty &\subset \text{Ch } \mathcal{M}, \\ \{(c_n^{(1)}, \dots, c_n^{(\ell)})\}_{n=1}^\infty &\subset (\mathbb{R}^+)^{\ell}, \end{aligned}$$

such that:

$$\begin{cases} \lim_{n \rightarrow \infty} c_n^{(j)} = \infty, & (j = 1, \dots, \ell), \\ \lim_{n \rightarrow \infty} (x_n^{(0)}, x_n^{(1)} c_n^{(\hat{J}_1^c)}, \dots, x_n^{(\ell)} c_n^{(\hat{J}_\ell^c)}; \xi_n^{(0)} c_n, \xi_n^{(1)} c_n^{(\hat{J}_1^c)}, \dots, \xi_n^{(\ell)} c_n^{(\hat{J}_\ell^c)}) \\ \quad = (x^{(0)}, 0, \dots, 0; 0, \xi^{(1)}, \dots, \xi^{(\ell)}), \end{cases}$$

where $c_n := \prod_{j=1}^\ell c_n^{(j)}$, $\hat{J}_j^c := \{1, \dots, \ell\} \setminus \hat{J}_j$, and $c_n^{(J)} := \prod_{j \in J} c_n^{(j)}$ for any $J \subseteq \{1, \dots, \ell\}$. In particular we have $\lim_{n \rightarrow \infty} (x_n^{(1)}, \dots, x_n^{(\ell)}, \xi_n^{(0)} c_n) = (0, \dots, 0, 0)$. Since $(\xi^{(1)}, \dots, \xi^{(\ell)}) \neq 0$, we may assume that

$$t_n := |(\xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)})| > 0.$$

We consider the sequence $\{(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}; t_n^{-1}(\xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)}))\}_{n=1}^\infty \subset \text{Ch } \mathcal{M}$. By extracting subsequence, we may assume that there exists $\zeta_0 \neq 0$ such that

$$\lim_{n \rightarrow \infty} t_n^{-1}(\xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)}) = \zeta_0.$$

Choose $1 \leq k \leq \ell$ with $\lim_{n \rightarrow \infty} \xi_n^{(k)} c_n^{(\hat{J}_k^c)} = \xi^{(k)} \neq 0$. Since $\lim_{n \rightarrow \infty} c_n^{(j)} = \infty$, we have

$$\lim_{n \rightarrow \infty} \frac{\xi_n^{(0)}}{\xi_n^{(k)}} = \lim_{n \rightarrow \infty} \frac{\xi_n^{(0)} c_n}{\xi_n^{(k)} c_n^{(\hat{J}_k^c)} c_n^{(\hat{J}_k)}} = 0.$$

Therefore, we see that $\zeta_0 = (0, \zeta_{01}, \dots, \zeta_{0\ell}) \neq 0$. Hence

$$\begin{aligned} & \lim_{n \rightarrow \infty} (x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}; t_n^{-1}(\xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)})) \\ &= (x^{(0)}, 0, \dots, 0; \zeta_0) \in \dot{T}_Y^* X \cap \text{Ch } \mathcal{M}, \end{aligned}$$

which contradicts the non-characteristic condition. Thus $\dot{S}_\chi^* \cap C_{\chi^*}(\text{Ch } \mathcal{M}) = \emptyset$. Hence we obtain the desired result. \square

Example 5.5. Take a point $p = (q; \xi) \in \times_{X, 1 \leq j \leq \ell} T_{Y_j} \iota(Y_j)$, and set $x_0 := \tau(p) \in Y$. Recall the notation in Remark 1.12. Then under the assumption of Theorem 5.4, every solution to \mathcal{M} defined on $\text{Cone}_\chi(p, \epsilon)$ for some $\epsilon > 0$ extends automatically a solution defined on a full neighborhood of x_0 .

Next, we consider the real cases. Let M be a real analytic manifold, and $\chi = \{N_i\}_{i=1}^\ell \subset M$. Assume that each N_i and $N := \bigcap_{i=1}^\ell N_i$ are real analytic submanifolds of M . We consider the multi-normal deformation \widetilde{M}_χ along χ . Let X be the complexification of M , and Y the complexification of N in X . Let $\iota: M \hookrightarrow X$ the canonical embedding. Let \mathcal{B}_M be the sheaf of hyperfunctions on M . Then by (5.5) we obtain

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{B}_M)) &\rightarrow \tau^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{B}_M)) \otimes \omega_{N/M}^{\otimes -1} \\ &\rightarrow Rp_{1*}^+(p_2^+)^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{B}_M)) \otimes \omega_{N/M}^{\otimes -1} \xrightarrow{+1}. \end{aligned}$$

For any conic subset $A \subset T^*X$ we can define $\iota^\#(A) := T^*M \cap C_{T_M^* X}(A)$ ([7, Definition 6.2.3]). Note that $(x_0; \xi_0) \in \iota^\#(A)$ if and only if there exists a sequence $\{(x_\nu + \sqrt{-1}y_\nu; \xi_\nu + \sqrt{-1}\eta_\nu)\}_{\nu=1}^\infty \subset A$ such that

$$\lim_{\nu \rightarrow \infty} (x_\nu + \sqrt{-1}y_\nu; \xi_\nu) = (x_0; \xi_0), \quad \lim_{\nu \rightarrow \infty} |y_\nu| |\eta_\nu| = 0.$$

Theorem 5.6. Assume that $N \hookrightarrow M$ is hyperbolic for \mathcal{M} ; that is,

$$(5.7) \quad \dot{T}_N^* M \cap \iota^\#(\text{Ch } \mathcal{M}) = \emptyset.$$

Then

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{B}_M)) \simeq \tau^{-1} R\mathcal{H}om_{\mathcal{D}_Y}(Df^* \mathcal{M}, \mathcal{B}_N).$$

Proof. By (5.7), we see that Y is non-characteristic for \mathcal{M} on a neighborhood of N . By the non-characteristic division theorem, we have

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{B}_M)) \otimes \omega_{N/M}^{\otimes -1} \simeq R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{B}_N).$$

By Corollary 6.4.4 of [7], we have

$$\mathrm{SS}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)) \subset \iota^\# \mathrm{SS}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_M)) = \iota^\#(\mathrm{Ch} \mathcal{M}).$$

As in the proof of Theorem 5.4 we have

$$\mathrm{supp}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{B}_M))) \cap \dot{S}_\chi^* \subset \dot{S}_\chi^* \cap C_{\chi^*}(\iota^\#(\mathrm{Ch} \mathcal{M})) = \emptyset.$$

and we obtain the desired result. \square

We shall give another result. Let us take $\chi = \{N, M\}$ with $N \subset M \subset X$ (Takeuchi's case). Then $S_\chi = T_N M \times_M T_M X$ and $S_\chi^* = T_N^* M \times_M T_M^* X \simeq T_{N \times_M T_M^* X}^* T_M^* X$. Under the notation of [15], we set

$$\begin{aligned} \nu_{NM} &:= \nu_\chi: D^b(k_X) \rightarrow D^b(k_{T_N M \times_M T_M X}), \\ \nu\mu_{NM} &:= \nu_\chi^{\wedge 2}: D^b(k_X) \rightarrow D^b(k_{T_N M \times_M T_M^* X}), \\ \mu_{NM} &:= \mu_\chi: D^b(k_X) \rightarrow D^b(k_{T_N^* M \times_M T_M^* X}). \end{aligned}$$

As usual, let \mathcal{E}_X be the sheaf of *ring of microdifferential operators* on T^*X and $\{\mathcal{E}_X(m)\}_{m \in \mathbb{Z}}$ the usual *order filtration* on \mathcal{E}_X . Let U be a \mathbb{C}^\times -conic open subset of T^*X , and Σ a \mathbb{C}^\times -conic involutory closed analytic subset of U . Set $\mathcal{I}_\Sigma := \{P \in \mathcal{E}_X(1)|_U; \sigma_1(P)|_\Sigma \equiv 0\}$ and $\mathcal{E}_\Sigma := \bigcup_{m \in \mathbb{N}_0} \mathcal{I}_\Sigma^m$. Here $\sigma_m(P)$ denotes the *principal symbol* of $P \in \mathcal{E}_X(m)$, and we set $\mathcal{I}_\Sigma^0 := \mathcal{E}_X(0)|_U$. Namely, $\mathcal{E}_\Sigma \subset \mathcal{E}_X|_U$ is a subring generated by \mathcal{I}_Σ .

Definition 5.7. (1) Let U be a \mathbb{C}^\times -conic open subset of T^*X , and Σ a \mathbb{C}^\times -conic involutory closed analytic subset of U . Let \mathfrak{M} be a coherent \mathcal{E}_X -module defined on U .

(a) An \mathcal{E}_Σ submodule \mathfrak{L} of \mathfrak{M} is called an \mathcal{E}_Σ -*lattice* of \mathfrak{M} if \mathfrak{L} is $\mathcal{E}_X(0)$ -coherent and $\mathcal{E}_X \mathfrak{L} = \mathfrak{M}$.

(b) We say that \mathfrak{M} has regular singularities along Σ if for any $p^* \in U$, there exist an open neighborhood V of p^* and an \mathcal{E}_Σ -lattice $\mathfrak{L} \subset \mathfrak{M}|_V$.

(2) Let Σ be a \mathbb{C}^\times -conic involutory closed analytic subset of \dot{T}^*X , and \mathcal{M} a coherent \mathcal{D}_X -module. Then we say that \mathcal{M} has regular singularities along Σ if so does $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$.

We impose the

Condition 5.8. (1) $\Lambda \subset \dot{T}^*X$ is a \mathbb{C}^\times -conic closed regular involutory complex submanifold,

(2) \mathcal{M} has regular singularities along Λ ,

(3) $\dot{T}_N^*M \cap \iota^\#(\Lambda) = \emptyset$.

By Condition 5.8 (2), we have $\text{Ch } \mathcal{M} \subset \Lambda \sqcup \text{supp } \mathcal{M}$. Hence by virtue of Condition 5.8 (3), we see that Y is non-characteristic for \mathcal{M} on neighborhood of N . Let $\mathcal{D}b_M$ be the sheaf of distributions on M .

Theorem 5.9. Assume Condition 5.8. Then

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{D}b_M)) \simeq \tau^{-1} R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{D}b_N).$$

Proof. Consider

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{D}b_M)) &\rightarrow \tau^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{D}b_M)) \otimes \omega_{N/M}^{\otimes -1} \\ &\rightarrow Rp_{1*}^+(p_2^+)^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{D}b_M)) \otimes \omega_{N/M}^{\otimes -1} \xrightarrow{+1}. \end{aligned}$$

Under Condition 5.8, we have

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{D}b_M)) \otimes \omega_{N/M}^{\otimes -1} &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{D}b_M)) \otimes \omega_{N/M}^{\otimes -1} \\ &\simeq R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{D}b_N), \end{aligned}$$

and

$$\text{SS}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M)) \subset \iota^\#(\Lambda \sqcup \text{supp } \mathcal{M}).$$

As in the proof of Theorem 5.4 we have

$$\text{supp}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{D}b_M))) \cap \dot{S}_\chi^* \subset \dot{S}_\chi^* \cap C_{\chi^*}(\iota^\#(\Lambda \sqcup \text{supp } \mathcal{M})) = \emptyset,$$

and this entails that $Rp_{1*}^+(p_2^+)^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{D}b_M)) = 0$, and we obtain the desired result. \square

Theorem 5.10. *Assume Condition 5.8. Then*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{C}_M^\infty)) \simeq \tau^{-1} R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{C}_N^\infty).$$

Proof. Consider

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{C}_M^\infty)) &\rightarrow \tau^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{C}_M^\infty)) \otimes \omega_{N/M}^{\otimes -1} \\ &\rightarrow Rp_{1*}^+(p_2^+)^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{C}_M^\infty)) \otimes \omega_{N/M}^{\otimes -1} \xrightarrow{+1}. \end{aligned}$$

Under Condition 5.8, we have

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{C}_M^\infty)) \otimes \omega_{N/M}^{\otimes -1} &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{W}_{M,N}^\infty) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{C}_N^\infty), \end{aligned}$$

and

$$\mathrm{SS}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M^\infty)) \subset \iota^\#(\Lambda \sqcup \mathrm{supp}\mathcal{M}).$$

Here $\mathcal{W}_{M,N}^\infty$ is the sheaf of Whitney functions on N . Thus the proof is same as in Theorem 5.4. \square

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